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ON REPRESENTING AN INTEGER AS THE HARMONIC MEAN OF INTEGERS

SOLOMON W. GOLOMB, University of Southern California

1. Introduction and summary. In this note, we pose and answer the question: In what ways, and in how many ways, is a positive integer the harmonic mean of two positive integers? Equivalently, given the positive integer n, what are the ordered pairs (a, b) of positive integers such that $2/n = 1/a + 1/b$?

We show that the general solution of this equation is

$$\frac{2}{n} = \frac{1}{\frac{(u+v)n}{2u}} + \frac{1}{\frac{(u+v)n}{2v}} \tag{1}$$

where u and v are coprime divisors of: n if n odd; $n/2$ if n even. From this we show that the number of representations $h(n)$ of n as the harmonic mean of two positive integers is given by

$$h(n) = \begin{cases} \sum\limits_{d|n} 2^{\nu(d)}, & n \text{ odd} \\ \sum\limits_{d|n/2} 2^{\nu(d)}, & n \text{ even} \end{cases} \tag{2}$$

where $\nu(d)$ is the number of distinct prime divisors of d.

Finally, we discuss the relationship of these results to the more general "Egyptian fraction" problem [1], [2], of representing any rational number as a sum of reciprocals of distinct integers.

2. The general representation. Suppose $2/n = 1/a + 1/b$, where n, a, and b are all positive integers. If we rewrite this as $2/n = (2/n)\alpha + (2/n)\beta$, then we have rational numbers α and β with $\alpha + \beta = 1$, $0 < \alpha < 1$, $0 < \beta < 1$. Writing α and β over a common denominator m, so that $\alpha = u/m$ and $\beta = v/m$, we have $1 = \alpha + \beta = (u+v)/m$, so that $m = u + v$. Thus $\alpha = u/(u+v)$ and $\beta = v/(u+v)$. If there is a common divisor of u and v, it may be divided out of numerators and denominator without affecting the values of α and β. Thus it is no loss of generality to pick $(u, v) = 1$, in which case also $(u, u+v) = 1$ and $(v, u+v) = 1$.

We now have

$$\frac{2}{n} = \frac{1}{a} + \frac{1}{b} = \frac{2}{n}\alpha + \frac{2}{n}\beta = \frac{1}{\frac{(u+v)n}{2u}} + \frac{1}{\frac{(u+v)n}{2v}} \tag{3}$$

with $(u, v) = 1$. In order for the denominators to be integers, $2u$ and $2v$ must divide $(u+v)n$.

If n is odd, then $u + v$ is even, so that $a = ((u+v)/2)(n/u)$ and $b = ((u+v)/2)(n/v)$. This requires u and v to be divisors of n. Conversely, if u and v are any two coprime divisors of odd n, we readily verify that $a = ((u + v)/2)(n/u)$ and $b = ((u + v)/2)(n/v)$ are integers with $1/a + 1/b = 2/n$.

If n is even, u and v may both be odd, or they may be of opposite parity. In either case, we may write $a = (u + v)(n/2)/u$ and $b = (u + v)(n/2)/v$, where u and v are coprime divisors of $n/2$. It is easily verified, as in the case of odd n, that this is the most general representation.

3. Enumeration of the representations. The number $h(n)$ of representations of $2/n$ as a sum $1/a + 1/b$, where $n, a,$ and b are all positive integers, is seen from the preceding section to be the number of ways of picking a pair of coprime divisors u and v of m, where m equals n, if n is odd, and m equals $n/2$, if n is even.

The pairs (u, v) of coprime divisors of m are enumerated as follows. Pick any divisor d of m. The $\nu(d)$ different primes which divide d must be allocated between u and v, which can be done in $2^{\nu(d)}$ different ways. These primes are used to form u and v with the same exponents which they had in d. Thus different divisors d of m lead to different ordered pairs (u, v) of coprime divisors of m. The total number of such ordered pairs is accordingly

$$h(n) = \sum_{d|m} 2^{\nu(d)}, \tag{4}$$

which gives

$$h(n) = \begin{cases} \sum_{d|n} 2^{\nu(d)}, & n \text{ odd} \\ \sum_{d|n/2} 2^{\nu(d)}, & n \text{ even.} \end{cases} \tag{2}$$

Note that $h(n)$ is insensitive to *which* odd primes divide n, depending only on the exponents to which they occur in the factorization of n. Some simple inequalities on the size of $h(n)$ include

$$2\tau(m) - 1 \leqq h(n) \leqq n \tag{5}$$

where as before $m = \begin{Bmatrix} n \text{ if } n \text{ odd} \\ n/2 \text{ if } n \text{ even} \end{Bmatrix}$, and $\tau(m)$ is the total number of divisors of m and thus satisfies $\tau(m) = \sum_{d|m} 1 = \sum_{d|m} 1^{\nu(d)}$ for purposes of comparison with $h(n)$.

Examples.

1. $h(15) = \sum_{d|15} 2^{\nu(d)} = 2^{\nu(1)} + 2^{\nu(3)} + 2^{\nu(5)} + 2^{\nu(15)} = 1 + 2 + 2 + 4 = 9.$

The representations are:

$$\frac{2}{15} = \frac{1}{15} + \frac{1}{15} = \frac{1}{8} + \frac{1}{120} = \frac{1}{9} + \frac{1}{45} = \frac{1}{10} + \frac{1}{30} = \frac{1}{12} + \frac{1}{20}$$

$$= \frac{1}{120} + \frac{1}{8} = \frac{1}{45} + \frac{1}{9} = \frac{1}{30} + \frac{1}{10} = \frac{1}{20} + \frac{1}{12}.$$

2. $h(10) = \sum_{d|5} 2^{\nu(d)} = 2^{\nu(1)} + 2^{\nu(5)} = 1 + 2 = 3.$

The representations are:

$$\frac{2}{10} = \frac{1}{10} + \frac{1}{10} = \frac{1}{6} + \frac{1}{30} = \frac{1}{30} + \frac{1}{6}.$$

4. Generalizations. If $k/n = 1/a_1 + 1/a_2 + \cdots + 1/a_k$, where k, n, and $a_1, a_2, \cdots a_k$ are all positive integers, then we have a representation of n as the harmonic mean of k integers. If $0 < (k/n) < 1$, it is well known [1] that k/n can be represented as the sum of the reciprocals of k or fewer *distinct* integers. (This is sometimes referred to [2] as the problem of the "Egyptian fractions".)

If $3/n = 1/a + 1/b + 1/c$, we may rewrite this as $3/n = (3/n)\alpha + (3/n)\beta + (3/n)\gamma$ with $\alpha = u/(u+v+w)$, $\beta = v/(u+v+w)$, $\gamma = w/(u+v+w)$, and $(u, v, w) = 1$. This leads to

$$\frac{3}{n} = \frac{1}{\dfrac{(u+v+w)n}{3u}} + \frac{1}{\dfrac{(u+v+w)n}{3v}} + \frac{1}{\dfrac{(u+v+w)n}{3w}}. \tag{6}$$

In this case, however, it is not necessarily true that all three of u, v, and w divide n. For example, if $u = 1$, $v = 2$ and $w = 3$, we have $u + v + w = 6$, which is divisible by all three of u, v, and w, even though $(u, v, w) = 1$. However, it should be possible to consider all cases with sufficient care to obtain from (6) the general solution to

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

For many values of n, representations of the form $3/n = 1/a + 1/b$ exist. In fact, we can settle the question of such representations completely. First, if n is a multiple of 3, we are looking at the representations $1/r = 1/a + 1/b$ where $3r = n$. But these are the representations $2/2r = 1/a + 1/b$, of which there are:

$$h(2r) = \sum_{d|r} 2^{\nu(d)} \tag{7}$$

as detailed in the previous section.

Otherwise, $(3, n) = 1$, and we write

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} = \frac{3}{n}\cdot\frac{u}{u+v} + \frac{3}{n}\cdot\frac{v}{u+v} = \frac{1}{\dfrac{(u+v)n}{3u}} + \frac{1}{\dfrac{(u+v)n}{3v}} \tag{8}$$

with $(u, v) = 1$, $u + v$ divisible by 3, and n divisible by both u and v. For $n > 1$, it is always possible to find at least two divisors, u and v, of n, but it is *not* always possible to find two such divisors with $u + v \equiv 0 \pmod 3$. Since $(3, n) = 1$, we must find divisors u and v of n which are respectively $+1$ and -1 modulo 3. The solution $u = +1$ is always available, but for a divisor $v \equiv -1 \pmod 3$ of n to exist, it is clearly necessary and sufficient that *not all prime divisors of n are* $\equiv 1 \pmod 6$. The number of distinct unordered representations $3/n = 1/a + 1/b$ for the case $(3, n) = 1$ is thus equal to the number of choices of pairs of divisors u and v of n with $(u, v) = 1$ and $u \equiv +1 \pmod 3$, $v \equiv -1 \pmod 3$. The reader is invited to obtain an explicit expression for this number of representations in terms of the canonical factorization of n.

A famous conjecture of Erdös and Strauss [3] asserts that for every positive integer n, $4/n$ can be written as a sum of the reciprocals of three or fewer distinct integers. Analogously to (3) and (6), we can rewrite $4/n = 1/a + 1/b + 1/c$ in the form

$$\frac{4}{n} = \frac{1}{\frac{(u+v+w)n}{4u}} + \frac{1}{\frac{(u+v+w)n}{4v}} + \frac{1}{\frac{(u+v+w)n}{4w}} \tag{9}$$

with $(u, v, w) = 1$. For even n, the conjecture is trivially true, since $4/n$ reduces. For odd n, we see that $u + v + w$ must be a multiple of 4, and that each of u, v, and w must divide $(u + v + w)n$.

The Strauss-Erdös conjecture has not yet been settled, though it has been verified to very large values of n. It has also been generalized (cf. [3]) as follows: Let k be a fixed positive integer. Then there is a number $n_0 = n_0(k)$ such that for all $n > n_0(k)$, the fraction k/n can be written as the sum of the reciprocals of three or fewer distinct integers. As before, we may rewrite $k/n = 1/a + 1/b + 1/c$ in the form

$$\frac{k}{n} = \frac{1}{\frac{(u+v+w)n}{ku}} + \frac{1}{\frac{(u+v+w)n}{kv}} + \frac{1}{\frac{(u+v+w)n}{kw}}, \tag{10}$$

with $(u, v, w) = 1$. At very least, the representations (9) and (10) simplify the computational process of verification or nonverification of these conjectures.

This research was supported in part by the Air Force Office of Scientific Research under Grant Number AFOSR-68-1555.

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1. S. W. Golomb, An algebraic algorithm for the representation problems of the Ahmes papyrus, Amer. Math. Monthly, 69 (1962), 785–787.

2. M. N. Bleicher, A new algorithm for the expansion of Egyptian fractions, J. Number Theory, 4 (1972), 342–382.

3. W. Sierpinski, A Selection of Problems in the Theory of Numbers, Pergamon Press/PWN Polish Scientific Publishers, Warszawa, 1964; problems P^2_{62} and P^2_{63}, page 113.

THE KLEIN BOTTLE AS AN EGGBEATER

RICHARD L. W. BROWN, York University, Toronto, Canada.

1. The Klein bottle. The Moebius band is a one-sided surface that is well known outside the field of topology. It has been the subject of parlor tricks ([1] page 259) and works of art ([2] pages 40 and 41). It is defined as the surface formed from a number of distinct unordered representations $3/n = 1/a + 1/b$ for the case $(3, n) = 1$ is thus equal to the number of choices of pairs of divisors u and v of n with $(u, v) = 1$ and $u \equiv +1 \pmod 3$, $v \equiv -1 \pmod 3$. The reader is invited to obtain an explicit expression for this number of representations in terms of the canonical factorization of n.

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The Moebius Band

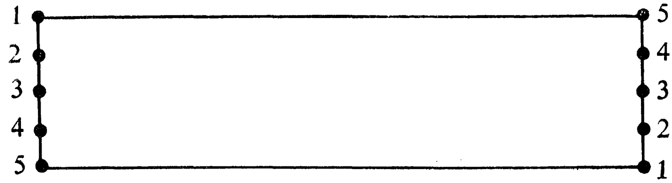

FIG. 1(a)

The Moebius Band

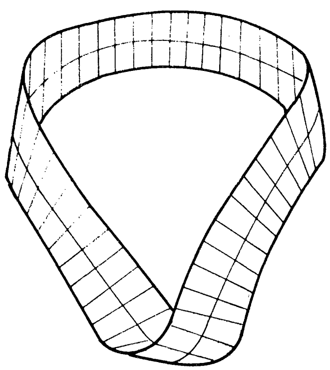

FIG. 1(b)

The Moebius Band

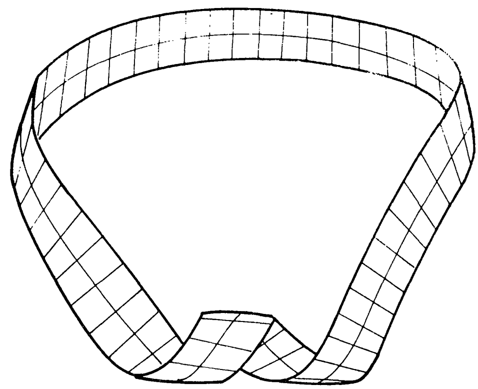

FIG. 1(c)

rectangular strip by identifying two opposite edges in the manner indicated in Fig. 1(a). This joining of opposite edges can be carried out in 3-space by bringing the ends together and matching them after making a half twist. The resulting surface is shown in Fig. 1 (b). Note that the same join can be achieved by giving the strip, for example, three half twists. Thus Fig. 1(c) is also a model of the Moebius band in 3-space.

The Klein Bottle

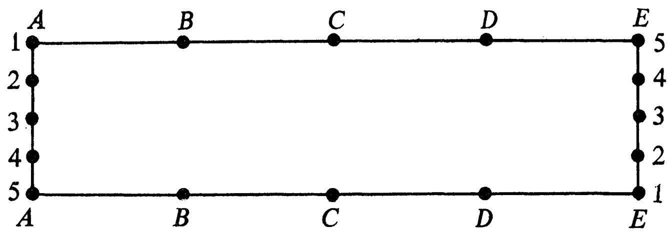

FIG. 2(a)

The Klein bottle is a one-sided surface like the Moebius band, but, unlike the Moebius band, it has no edges. The Klein bottle K^2 is defined as the surface obtained from a rectangular strip by identifying the ends with a half twist, and also identifying the top and bottom with no twist. This is shown in Fig. 2(a). To see why the Klein

The Klein Bottle

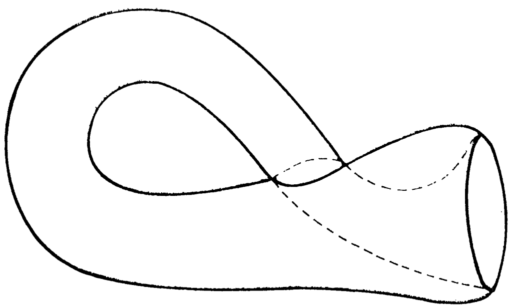

FIG. 2(b)

The Klein Bottle

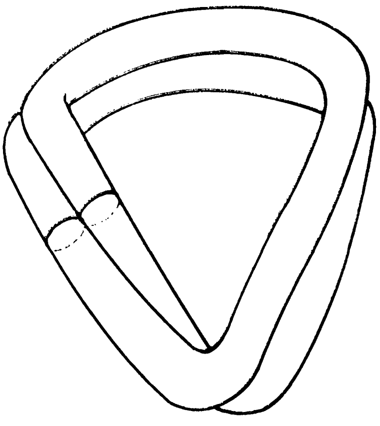

FIG. 2(c)

bottle is not so well known to the layman, let us try to carry out these joins in 3-space. Suppose we first join the top and bottom edges of the rectangle to form a tube, whose ends are to be matched in a certain way. The only apparent way to match the ends is to pass the tube through itself first and then connect the ends. The resulting object is sketched in Fig. 2(b) and this is the usual picture given for K^2. We emphasize that the circle of self-intersection is a flaw in the model, not a property of K^2 itself. We can obtain another (imperfect) model of K^2 as follows. First join the ends of the rectangle to form a Moebius band as in Fig. 1(b). Now join the top and bottom edges. Again it appears that a self-intersection must occur as we attempt the second join. (Try it with a strip of paper.) If we draw the center line on the Moebius band and bend the top and bottom edges until they are near each other but on opposite sides of the center line, we can join them through the center line. The resulting object, sketched in Fig. 2(c), resembles a Moebius band made from a double-barrelled hose with a figure-8 cross-section. This object is more easily constructed from a paper strip than the usual one in Fig. 2(b). (Try it!) Note that in 2(b) the self-intersection is locally two untwisted bands meeting in a circle whereas in 2(c) the self-intersection is locally two Moebius bands meeting in a circle.

To really understand K^2, one must imagine, in looking at Figs. 2(b) and 2(c), that no self-intersections occur. Otherwise one has an erroneous conception; for if plastic models of either figure were built, the models would have two sides, an inside and an outside, and, for example, it would be difficult to paint the inside! A true model of K^2 in this sense is impossible, for it is a standard result in algebraic topology (for example [3] page 179) that K^2 will not fit in 3-space without self-intersections. However K^2 will fit in n-space provided $n \geqq 4$. We will give formulas shortly. But first we wish to meet the obvious objection that models in n-space for $n > 3$ could not arise in everyday life and have a "reality" that is purely mathematical, understandable only by people with higher dimensional intuition. In the next section we construct, from equipment available in nearly every household, a model representing K^2 in 5-space R^5, a model concrete enough to hold in your hand.

2. The eggbeater model. The notion of the configuration space of a mechanical system has proved basic in the study of mechanics. (See [4], page 509.) It is simply the set of all possible positions of the system. If the system is just a particle moving in 3-space, then the configuration space is R^3. Two particles in 3-space require three coordinates to specify the position of the first particle, and three more for the second. Hence the configuration space here is R^6; or, if the particles cannot occupy the same place, the configuration space is $R^6 - \Delta$ where Δ is the set of all points of the form (x, y, z, x, y, z). Thus we have a way of thinking of a high dimensional space in terms of a concrete mechanical system in ordinary 3-space.

Our model M is a mechanical system constructed from an eggbeater and two beads. One bead is attached to the handle, and the other is threaded onto the beater on which it is free to move. In Fig. 3 we show the specific model we have in mind, with the beads being points A and B.

The Klein Bottle in R^5

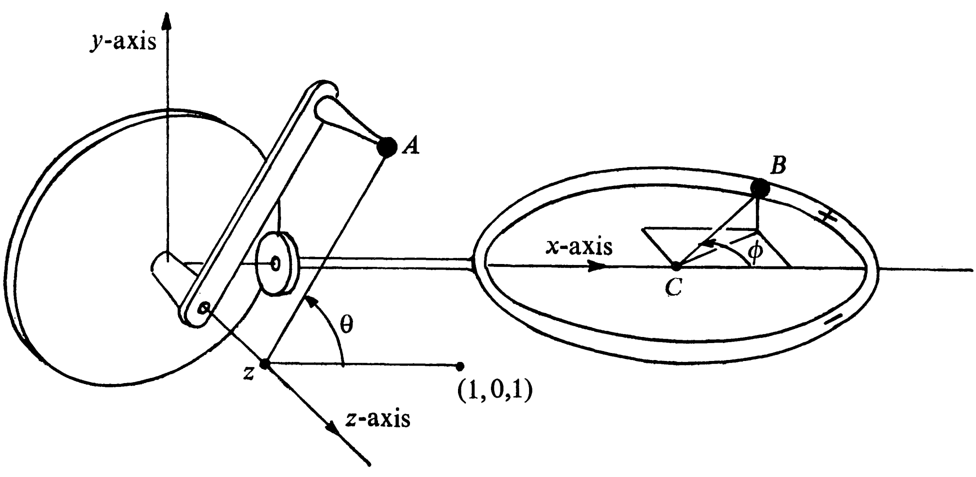

FIG. 3

Point A moves in a circle of unit radius with center $Z = (0, 0, 1)$ in the plane $z = 1$. Point B is free to move in a unit circle with center $C = (c, 0, 0)$. The plane of this circle is not fixed. It contains the x-axis and rotates about the x-axis as A is moved. This action is accomplished by the usual eggbeater gearing, about which we make the following assumption: two complete rotations of the handle should produce an odd number of complete rotations of the beater.

Let the coordinates of A be $(x_1, y_1, 1)$ and of B be (x_2, y_2, z_2). Then the 5-tuple $(x_1, y_1, x_2, y_2, z_2)$ in R^5 determines the position of both points, although not every 5-tuple arises in this way. Hence the set M of all positions of A and B can be considered as a subset of R^5.

A physicist might say our system has two degrees of freedom which would suggest to a mathematician that we are dealing with a surface in R^5. To be more precise, let us introduce angular coordinates. Let θ denote the angle between ZA and the line from Z to $(1,0,1)$ measured in the range $-\pi \leqq \theta \leqq \pi$. The x-axis divides the beater into two semicircular pieces. Choose one piece and mark it permanently as the positive side. Let ϕ denote the angle between CB and the positive x-axis measured in the range $-\pi \leqq \phi \leqq \pi$, where ϕ is positive when B is on the positive side of the beater. Let S denote the square $\{(\theta, \phi): -\pi \leqq \theta \leqq \pi, -\pi \leqq \phi \leqq \pi\}$ in R^2. See Fig. 4. For each $(\theta, \phi) \in S$ we let $P(\theta, \phi)$ be the point in R^5 obtained by putting A in the position θ and B in the position ϕ. Then clearly P maps S onto $M \subset R^5$. Looking more closely at Fig. 3, we see that P maps the interior of S injectively; that is, if $-\pi < \theta < \pi$ and $-\pi < \phi < \pi$ then $P(\theta, \phi) = P(\theta', \phi')$ implies that $\theta' = \theta$ and $\phi' = \phi$. Along the edges of S, P is not injective, and certain identifications are made.

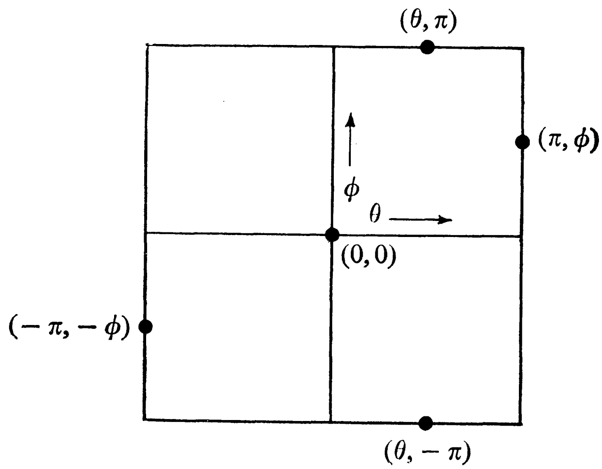

FIG. 4

Clearly, from Fig. 3, $P(\theta, -\pi) = P(\theta, \pi)$ for $-\pi \leqq \theta \leqq \pi$. These points are shown in Fig. 4. Furthermore, $P(\pi, \phi) = P(-\pi, -\phi)$ for $-\pi \leqq \phi \leqq \pi$ because as A is moved from π to $-\pi$, the beater makes an odd number of half turns and hence the positive side takes the position previously occupied by the negative side. These points are also shown in Fig. 4. There are no other identifications made by P. However, a rectangle with these identifications is precisely K^2. Hence P maps S onto a copy of K^2, namely M, in R^5.

3. Some formulas. We can express the coordinates of A and B in terms of θ and ϕ as follows:

$$A = (\cos\theta, \sin\theta, 1)$$

$$B = \left(c + \cos\phi, \sin\phi \cos\frac{2n+1}{2}\theta, \sin\phi \sin\frac{2n+1}{2}\theta\right)$$

(We are assuming the beater is in the xy-plane when $\theta = 0$ and that 2 turns of the handle produce $2n + 1$ turns of the beater.) This gives the following formula for the function P:

$$P(\theta, \phi) = \left(\cos\theta, \sin\theta, c + \cos\phi, \sin\phi \cos\frac{2n+1}{2}\theta, \sin\phi \sin\frac{2n+1}{2}\theta\right).$$

However, if all we want is a function from S to R^5 having the correct identifications, we can simplify the above function by taking $c = 0$, $n = 0$. This gives

(1) $$Q(\theta, \phi) = \left(\cos\theta, \sin\theta, \cos\phi, \sin\phi\cos\frac{\theta}{2}, \sin\phi\sin\frac{\theta}{2}\right)$$

as an embedding of K^2 in R^5.

It is also possible to embed K^2 (indeed, any surface) in R^4. To find a formula, let us first work with the close relative of K^2, the torus T^2. We define T^2 as a rectangle with opposite pairs of edges identified with no twists. Clearly then the function

(2) $$f(\theta, \phi) = (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$$

produces exactly the right identifications on the square S to give us an embedding of T^2 in R^4. However there is a well known embedding of T^2 in R^3. (The tire industry has heavily exploited this fact!) We can use the sketch in Fig. 5 to read off a formula:

(3) $$g(\theta, \phi) = ((2 + \cos\phi)\cos\theta,\ (2 + \cos\phi)\sin\theta,\ \sin\phi).$$

The Torus in R^3

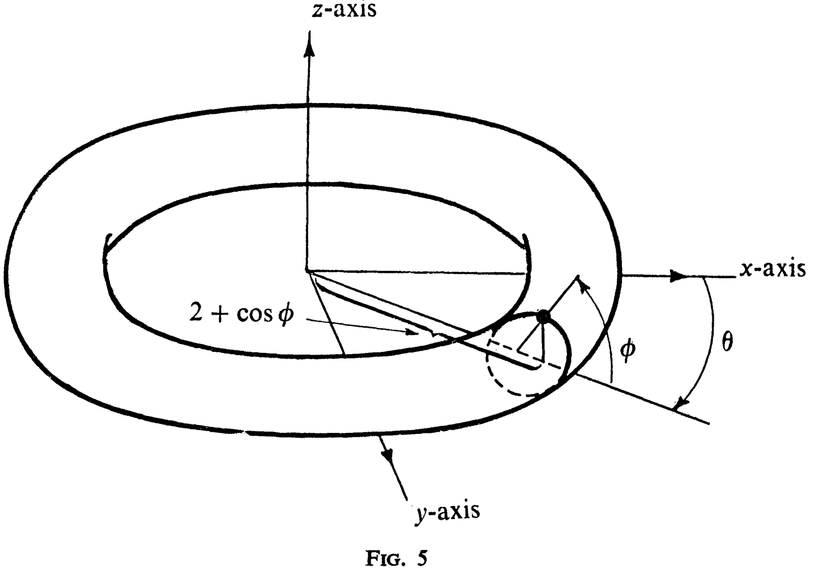

FIG. 5

If we compare (2) with (3), we are led to changing (1) to

(4) $$h(\theta, \phi) = \left((2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi\cos\frac{\theta}{2}, \sin\phi\sin\frac{\theta}{2}\right).$$

This function can be checked directly as having the correct identifications to give an embedding of K^2 in R^4. Formula (4) implies there is a mechanical model consisting, for example, of two points moving in the plane whose configuration space is K^2. But the author, lacking the gift of mechanical ingenuity, was unable to discover one.

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BEGINNING STATISTICS AT THE TRACK

W. P. COOKE, University of Wyoming

It is traditional for teachers of introductory statistics and probability to use gambling games in the early going as vehicles for discussing probability and expectation. The more familiar casino-type games like Roulette, Chuck-a-Luck, and Craps are usually employed, and the expected value computations serve to exhibit why one should lose in the long run.

Certainly it would be a more motivating experience for beginning students if they could see examples of *winning* gambling strategies based solely upon elementary statistical notions. One approach the author has found to be popular in the classroom is presented herein. The setting is the greyhound track rather than the casino, but the essential computations are the same. Further, the notions of regularity and variability are graphically evident in the data used.

Casino vs. track. The disadvantage of casino games which utilize mechanical devices is that the wagers are all made with negative expected gain. While the probabilities involved, interpreted as relative frequencies, are easy to determine, the odds given by the house are not fair odds.

At the horse or dog track, however, the bettor finds himself in a different situation. The probability that a given animal will win a given race, or more accurately, the relative frequency with which an animal like this may be expected to win in races like this, is extremely difficult to assess. The odds on that animal, however, are not fixed in advance by a "house", but rather are determined by the proportionate number of "win tickets" purchased on the animal. In pari-mutuel wagering all money bet to win goes into a pool, a fixed percentage (usually from 15 to 20 percent) is removed for taxes, expenses, and profits to the track, and the remainder, called

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Assume for the moment that one has correctly decided that the probability a certain greyhound will win a race is 1/4, but for some reason the bettors have overlooked him to the extent that he "closes" at final odds of 6 to 1. Then a wager can be made with *positive* expected gain.

Professional gamblers call that situation an *overlay*, and it is well known amongst track habitués that an overlay strategy is a winning strategy. Only the professional, however, whose experience at the track enables him to estimate closely the required probability, can use such a strategy.

Overlays for the amateur. By an "amateur" at the track we mean a bettor who has not learned how to utilize data on past performances of racing animals to estimate their probabilities of winning. He does, however, understand the relative frequency interpretation of a probability and the concept of expectation.

This amateur does have access to some "professional" opinions. There are students of racing whose selections are *published*. Examples are the analysts whose selections make up the "Consensus" in *The Daily Racing Form*, various newspaper selectors, and "tip-sheets" on sale at the track.

What the amateur can do easily is keep records on how well a certain *selector* performs. If, for example, the first-ranked dog of Selector X has been winning about 1/4 of the races, then an overlay strategy could be, "Wager on Selector X's first-ranked dog only if it closes at odds of 4 to 1 or better." This strategy is designed to yield a minimum 25% return on investment.

The analogy here is an actuarial one. The amateur is not concerned about which animal produces a winning ticket, but only that the future relative frequency of winners does not differ appreciably from 1/4.

An example. In the Denver, Colorado area there are many greyhound tracks, with contiguous seasons running from the middle of January until the end of November. Further, the *Rocky Mountain News* [2], a newspaper published daily, carries a section titled "Staff (or Springs) Selections" giving first, second, and third-ranked selections for each of the 11 races that are run nightly except for Sundays.

The author has been keeping track of the frequency of winners that were first-ranked Staff selections since April 1971. From April 1971 through August 1971, for Staff selections made for Cloverleaf Kennel Club at Loveland, Colorado and Mile High Kennel Club (MHKC) at Commerce City, Colorado, the relative frequency of first-ranked winners had been a little better than 1/5. Note that there are almost always 8 dogs in a greyhound race, so "Staff" at least picks winners at a better rate than the 1/8 produced by a random selection scheme.

In September 1971 the season began at Rocky Mountain Kennel Club (RMKC) at Colorado Springs. It was decided to try as a strategy, "Wager on any first-ranked Springs selection who closes at odds of 5 to 1 or better."

It is clear that if the probability of winning were to remain at 1/5 and the odds were always exactly 5 to 1, then 20% should be the rate of return on investment. Since, however, one would occasionally wager at odds better than 5 to 1, 20% should be the *minimum* return-rate expected for the strategy.

The results for a 55-day experience (but only 530 races; sometimes the newspaper did not publish complete results) using this strategy at RMKC in 1971 follow. There was a total of 146 first-ranked selections who closed at 5 to 1 or better. Using $10 as the unit wager the total investment was $1,460. The actual profit was $471, a return-rate of 32%. The true relative frequency of first-ranked winners was 110/530, about 1/4.8, or a little better than but very near the estimated 1/5.

Impressive? Perhaps, but there is of course much more that the reader might want to know about the performance of this strategy than just overall profit, although that is of course the final measure of its validity. Figure 1 on p. 253 shows the daily relative frequencies of first-ranked "Staff" winners, along with the cumulative relative frequency for a 77 day experience at MHKC and RMKC in 1971. That figure should exhibit nicely the meaning of variability and statistical regularity, as well as to serve notice that a "long run" strategy, based solely upon positive expectation, requires a long run experience for dependability.

Notice that the *daily* relative frequency of winners varies, with no discernible pattern, between 0 and .54 (6/11 winners), while the *cumulative* relative frequency seems to be trying to settle down to about 1/5. Thus the "once-a-year" bettor would not be too impressed by our long-run strategy, but the "daily" bettor might do well to consider using it, particularly if he has been a consistent loser.

It is not just how closely the relative frequency estimates the "true" probability, however, that the prospective bettor should consider. Rather he should be more concerned about the kind of fluctuations to expect in his *bankroll* as he daily pursues this strategy. Consequently, Table 1 is shown for a 34-day experience in early 1972 at the Interstate Kennel Club (IKC) at Byers, Colorado, using the same strategy as before. Further, columns are included to show what happened if one merely had wagered on all first-ranked Staff selections *regardless* of odds.

Table 1 exhibits nicely the kind of experience a bettor might expect with daily use of our strategy. There are violent fluctuations in his bankroll, but the general trend of profit is upward. Further, to demonstrate the advantage to be gained in the educated rather than indiscriminate use of published selections, note that one would have experienced a small *loss* if he had not specified the recommended odds before placing a bet. Finally, observe that our "5 to 1" strategy may be in reality a "break-even" strategy when the expenses of daily track-going are included, although it can easily be made profitable by increasing the unit wager from $10 to $20 or $30 per bet.

Conclusion. Both Figure 1 and Table 1 are good classroom examples to help

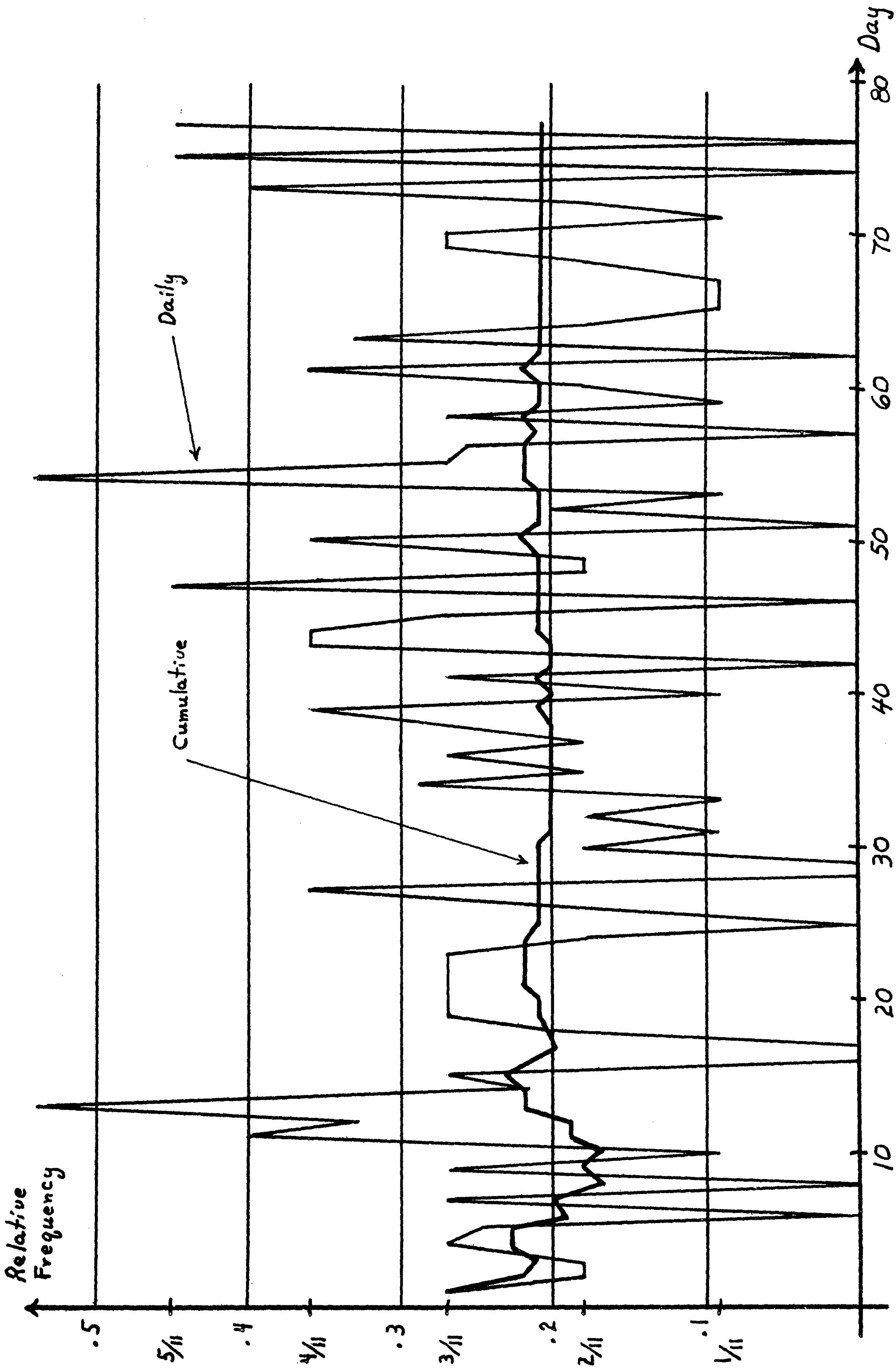

FIG. 1. Daily and Cumulative Relative Frequencies (MHKC and RMKC)

		No Odds Requirement				*5 to 1 Required*			
Day	*Date*	*Races*	*Won*	*Profit*	*Cum. Profit*	*Bets*	*Won*	*Profit*	*Cum. Profit*
1	1−13	11	2	−30	−30	3	0	−30	−30
2	1−14	5	2	+98	+68	3	1	+60	+30
3	1−15	11	1	−49	+19	4	1	+21	+51
4	1−17	11	5	+147	+166	3	1	+101	+152
5	1−18	11	0	−110	+56	2	0	−20	+132
6	1−19	11	3	+65	+121	5	2	+76	+208
7	1−20	11	2	−44	+77	5	0	−50	+158
8	1−21	5	2	+18	+95	2	0	−20	+138
9	1−22	11	1	−62	+33	3	0	−30	+108
10	1−24	11	1	−79	−46	1	0	−10	+98
11	1−25	11	2	−33	−79	2	0	−20	+78
12	1−26	11	3	−16	−95	4	0	−40	+38
13	1−27	11	2	−15	−110	1	0	−10	+28
14	1−28	6	1	+25	−85	3	1	+55	+83
15	1−29	11	0	−110	−195	2	0	−20	+63
16	1−31	11	3	+43	−152	6	2	+64	+127
17	2−2	11	1	−62	−214	6	0	−60	+67
18	2−3	11	2	+19	−195	7	2	+59	+126
19	2−4	3	1	+52	−143	2	1	+62	+188
20	2−5	11	2	+52	−91	4	2	+122	+310
21	2−7	11	2	−66	−157	3	0	−30	+280
22	2−8	11	1	+25	−132	7	1	+65	+345
23	2−10	11	3	+82	−50	7	2	+79	+424
24	2−11	5	0	−50	−100	3	0	−30	+394
25	2−12	11	1	−84	−184	5	0	−50	+344
26	2−14	11	3	+13	−171	6	1	+12	+356
27	2−15	11	2	+19	−152	7	2	+59	+415
28	2−16	11	0	−110	−262	6	0	−60	+355
29	2−17	11	4	+82	−180	5	2	+84	+439
30	2−18	3	1	+4	−176	0	0	0	+439
31	2−19	11	2	−22	−198	3	0	−30	+409
32	2−21	11	2	+22	−176	4	1	+30	+439
33	2−22	11	3	+39	−137	4	1	+24	+463
34	2−23	11	1	−41	−178	6	1	+9	+472
Totals		335	61	−178		133	24	+472	
		$61/335 \doteq 1/5.49 \doteq .18$				$24/133 \doteq 1/5.54 \doteq .18$			

TABLE 1.

IKC (1972): First-ranked "Staff Selection" illustrate notions of relative frequency, statistical regularity, and variability. Note that Figure 1 is a nice counterpart to the usual "coin-flipping" example wherein the relative frequency of heads is observed to be settling down to about 1/2 after many flips of the coin.

There is of course much more in the way of applied statistics that can be done with data on past performances of the animals themselves. The author assumes, for example, that a factor analysis could lead to good information about how to select winners. The main point of this paper, however, is to exhibit a strategy that is not commonly used by "average bettors," but that is very easy to understand and requires minimal record-keeping.

If the reader wishes to verify the examples, he is referred to the same source used by the author: back issues of [2] on hand in the Denver Public Library. By taking the time to list each day's Staff selections and then by reading the results which also appear in that newspaper, the reader may reconstruct the daily experiences ussd as examples herein.

Finally, for more examples of a similar nature, the reader is referred to Cooke [1]. There he will also find advice on modifications that will improve profit if one actually participates in the wagering at some track, and suggested statistical procedures that might be helpful.

Acknowledgement. The author wishes to acknowledge two comments by a referee of this paper. That referee astutely observed that if the strategy herein were to be widely adopted, the number of opportunities to make a wager at the specified odds would be reduced, since the increased number of bettors would drive down the odds. The author feels that there is not much chance for that wide adoption to occur, since the strategy is much more conservative than strategies favored by nonprofessional bettors.

Another comment pointed out the implied (by Table 1) but not specifically mentioned assumption that the probability of a "Staff Selection" winning remains 1/5 regardless of the closing odds. The data does support that assumption, but only for that particular selector. The author has discovered that the public apparently prefers other selectors to "Staff." In [1] are suggested some modifications to use in case the particular selector one uses has a large following, thus generating lower odds.

Presented February, 1972 at winter meeting of Colorado-Wyoming chapter of American Statistical Association.

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MINIMUM TRIANGLES INSCRIBED IN A CONVEX CURVE

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1. Introduction. The following result, concerning the smallest circle containing copies of all closed curves of given length, has received attention from time to time (see, for example [1], [2]):

THEOREM 1. *Any plane closed curve of length L can be covered by a circular disk of radius $L/4$.*

A generalization to higher dimensions is proved in [1] and [2], and a number of references to earlier proofs are given in [1].

Observe that the smallest circle containing a planar curve must intersect the curve in a set of points whose convex hull contains the center of the circle. It follows that there is a triangle with vertices on the intersection of the curve with the circle and having the center of the circle in its interior (except in a degenerate case where the triangle reduces to a line segment). Thus, Theorem 1 is actually equivalent to the following theorem:

THEOREM 2. *A triangle inscribed in a circle and having the center in its interior has length at least twice the diameter.*

This theorem was posed as a problem in this MAGAZINE [3] where two proofs are given. It is not difficult to establish the following stronger result. *A triangle inscribed in a circle and having a given point P in its interior has length at least twice the minimum chord through P.*

In this paper, we shall prove the following generalization:

THEOREM 3. *If K is a plane convex curve and P a point interior to K, then any triangle inscribed in K and having P in its interior has length at least twice the minimum chord of K through P.*

As a corollary, we have the following:

COROLLARY 1. *If P be a point interior to the plane convex curve K and C a plane closed curve such that the convex hull of $C \cap K$ contains P, then C has length at least twice the minimum chord of K through P.*

This follows from Theorem 3, since if the convex hull of $C \cap K$ contains P, then some triangle with vertices in $C \cap K$ contains P.

It is of interest to compare Theorem 3 with a result of Laugwitz [4] that generalizes Theorem 2 to Minkowski planes (two dimensional Banach spaces). Laugwitz proved that a triangle inscribed in the unit circle of a Minkowski plane and containing the origin in its interior has length at least 4 (measured in the Minkowski metric). Expressed in terms of the Euclidean metric this becomes the following: Let K be a centrally symmetric convex curve and T a triangle inscribed in K with the center of K in its interior. Then the sum of the lengths of the sides of T, each divided by the corresponding parallel chord of K through the center, is at least 2. Observe that this implies Theorem 3 in case K is centrally symmetric and P its center, but our Theorem 3 does not imply the theorem of Laugwitz. It is worth remarking that this result of Laugwitz provides an analogue of Theorem 1 for the Minkowski plane. Namely, any closed curve of length L in a Minkowski plane can be covered by a translate of the unit disk of the given metric, magnified by the factor $L/4$.

Our proof of Theorem 3 depends on the case where K is a triangle, this case then following from certain properties of the pedal triangle that we treat in the next section. We prove Theorem 3 in Section 3 and devote Section 4 to a discussion of some conjectures about higher dimensional generalizations.

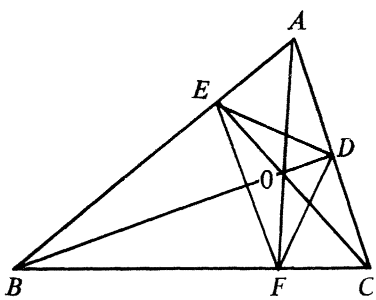

FIG. 1.

2. Properties of the pedal triangle. Consider an acute angled triangle ABC as in Figure 1, with pedal triangle DEF, so that D, E, F are the feet of the altitudes of ABC. Let a, b, c be the lengths of the sides of ABC opposite A, B, C respectively, and let R be the radius of the circumscribed circle of ABC. The point O is the orthocenter, i.e., the intersection of the altitudes. It is easily established that $\measuredangle AED = \measuredangle C$ and $\measuredangle ADE = \measuredangle B$, hence $ADE \sim ABC$. From the right triangle ADB we see that $AD = AB \cos A$, hence triangle ADE is obtained from similar triangle ABC by reducing by the factor $\cos A$. The circumscribed circle of ADE has AO as diameter, so that $AO = 2R \cos A$. From right triangle ADO, we then obtain $OD = 2R \cos A \cos C$. Similarly $OE = 2R \cos A \cos B$. These relations, found in Johnson [5], will be used to prove the following lemma:

LEMMA 1. *With the notation as in Figure* 1, *the perimeter of the pedal triangle is greater than twice the chord of triangle ABC through O parallel to DE.*

Proof. Let GH be the chord through O parallel to ED, with G on side AB and H on side AC. From the expressions derived above for OD and OE, we obtain

$$GH = GO + OH = \frac{2R\cos A\cos B}{\sin C} + \frac{2R\cos A\cos C}{\sin B}. \tag{1}$$

As shown in [5, p. 191], the perimeter of the pedal triangle is given by $2\Delta/R = (bc\sin A)/R$, where Δ is the area of ABC. To prove our lemma, we must show that this is greater than $2GH$. From the expression (1), we see that we must prove

$$4R^2\cos A\left(\frac{\cos B}{\sin C}+\frac{\cos C}{\sin B}\right) < bc\sin A, \tag{2}$$

or, using the fact that $bc = 4R^2 \sin B \sin C$,

$$\cos A\left(\frac{\cot B}{\sin^2 C}+\frac{\cot C}{\sin^2 B}\right) < \sin A. \tag{3}$$

Expressing $\sin^2 C$, $\sin^2 B$, and $\tan A$ in terms of $\cot B$ and $\cot C$, it is seen that (3) is equivalent to

$$(\cot B+\cot C)(1+\cot B\cot C) < \frac{\cot B+\cot C}{1-\cot B\cot C}; \tag{4}$$

an inequality that holds, since $\cot B \cot C < 1$. This completes the proof.

The preceding lemma shows that the perimeter of the pedal triangle is greater than twice any of the three chords through the orthocenter parallel to the sides of the pedal triangle. Now let P be any point interior to ABC. Then at least one of the three chords through P parallel to the sides of the pedal triangle has length less than or equal to the parallel chord through the orthocenter. Thus, we have the following lemma:

LEMMA 2. *Let P be any point interior to the acute angled triangle ABC. Then the perimeter of the pedal triangle is greater than twice the minimum chord of ABC through P.*

The familiar minimum property of the pedal triangle (see [6] p. 346) tells us that any triangle having its vertices on the respective sides of ABC has perimeter at least that of the pedal triangle. Hence, we have the following from Lemma 2:

LEMMA 3. *Let P be any point interior to the acute angled triangle ABC, and let triangle RST have its vertices on the respective sides of ABC. Then the perimeter of RST is greater than twice the minimum chord of ABC through P.*

In case ABC is not acute angled, we make the convention that "pedal triangle of ABC" shall mean the shortest altitude of ABC traversed twice, a degenerate triangle. Then we still retain the minimizing property of the pedal triangle [6, p. 351]. In other words, if ABC is not acute angled, and RST has its vertices on the respective sides of ABC, then the perimeter of RST is at least that of the shortest altitude of ABC traversed twice. Note also that in this case the minimum chord through any interior point P has length at most that of the shortest altitude (look at a chord parallel to the shortest altitude). Thus the perimeter of RST in this case is still greater than twice the minimum chord of ABC through P.

3. Proof of Theorem 3. Let the triangle RST be inscribed in K with P interior to

RST. Suppose K has supporting lines at R, S, and T respectively, forming the sides of a triangle ABC enclosing K, as shown in Figure 2. By Lemma 3, and in case ABC is not acute angled, by the remarks following Lemma 3, the perimeter of RST is greater than twice the minimum chord of ABC through P, hence greater than twice the minimum chord of K through P.

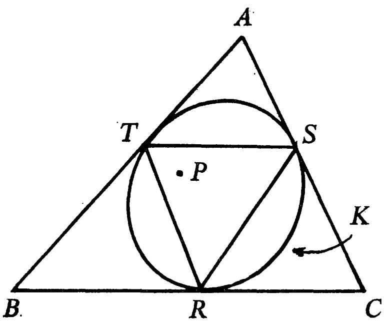

FIG. 2.

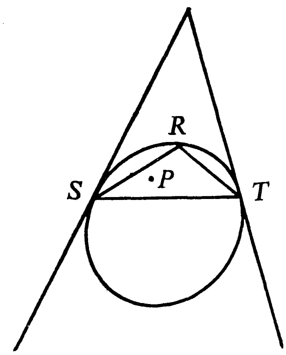

FIG. 3.

In case we cannot find such supporting lines forming a triangle enclosing K, then we have a situation as illustrated in Figure 3 (or possibly we may have to choose parallel supporting lines at S and T, say).

In this case, we see that the perimeter of RST is greater than twice the chord through P parallel to ST, hence the theorem holds also in this case. This completes the proof.

4. Open problems. If S is a closed and bounded subset of Euclidean n-space, and u a unit vector, then the *width of S in the direction u*, denoted by $b(S,u)$, is defined as the distance between the supporting hyperplanes of S orthogonal to direction u. The *mean width* of S is obtained by averaging $b(S,u)$ over all directions u. In other words, letting ω_n be the volume of the unit n-ball, so $n\omega_n$ is the surface area of its boundary the unit $(n-1)$-sphere, we have the mean width $B(S)$ of S defined as

$$B(S) = \frac{1}{n\omega_n} \int b(S,u)\,du, \tag{5}$$

where the integration is over the unit $(n-1)$-sphere.

In case $n = 2$ and K is a plane convex curve, a theorem of Cauchy asserts that the perimeter of K is exactly $\pi B(K)$. Thus Theorem 3 may be reformulated with "mean width" replacing "length", keeping in mind that if S is a line segment, $\pi B(S)$ is twice the length of the line segment. Thus we can say that if P is an interior point of a plane convex curve, then the mean width of triangles inscribed in K and containing P is minimized by a chord containing P (a degenerate triangle). In view of this, the following conjecture seems plausible:

CONJECTURE 1. *If K be a convex body in Euclidean n-space and P a point interior to K, then any simplex inscribed in K and having P in its interior has mean width greater than that of the minimum chord of K through P.*

It would be of interest to settle this even in the special case where K is a ball and P its center. In other words, we may venture the following special case:

CONJECTURE 2. *Any simplex inscribed in an n-ball and containing the center has mean width greater than that of a diameter.*

Of course Theorem 2 settles this last conjecture in case $n = 2$. If Conjecture 2 were true, then the following conjecture would also be correct:

CONJECTURE 3. *Any closed and bounded subset S of Euclidean n-space of mean width B can be covered by a ball whose diameter has mean width B.*

In other words, we venture that the size of the minimum ball containing a set of given mean width is maximized when that set is a line segment.

A computation shows that if S is a line segment of length $2R$ in Euclidean n space, then $B(S) = 4R\omega_{n-1}/n\omega_n$. Further, it can be shown that the mean width of a closed curve C of length $L(C)$ in n-space satisfies the inequality

$$B(C) \leqq \frac{\omega_{n-1}}{n\omega_n} L(C). \tag{6}$$

Using these facts, we see that Conjecture 3 implies that a closed curve of length L in n-space can be covered by a ball of radius $L/4$. This would provide a new proof of the generalization of Theorem 1 to n-space (which as we mentioned above is known to be true).

We also obtain plausible conjectures in higher dimensions when "mean width" is replaced by "total edge length". For example, analogous to Conjecture 1 we have

CONJECTURE 4. *If K be a convex body in Euclidean n-space and P a point interior to K, then any simplex inscribed in K and having P in its interior has total edge length greater than n times the length of the minimum chord of K through P.*

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THE COMPACTNESS THEOREM IN MATHEMATICAL LOGIC

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A topological space is said to be *compact* if every open covering contains a finite subcovering—or equivalently, if an intersection of closed sets is empty, then there must have been some finite intersection which was already empty. In logic the compactness theorem is sometimes stated in the following form: *If a (mathematical) theory is inconsistent, some finite subtheory must be inconsistent.* Because a proof is finite in length, the theorem follows from the fact that if a contradiction can be derived in a theory, there must be a finite set of axioms which imply both A and not A.

If, in addition, we restrict ourselves to those theories which are called *first order* or *elementary* (these terms are explained below) and assume a form of Gödel's completeness theorem—*A first order theory is consistent if and only if it has a model*—then the Compactness Theorem can be stated in the following more useful form:

A first order theory has a model if each of its finite subtheories has a model.

To fully understand this statement it is necessary to understand the notions of a *first order (elementary) theory*, a *finite subtheory*, and a *model*. Any mathematical theory is constructed within a *formal language*. The symbols of the language include logical symbols (individual variables, logical connectives, predicates, and quantifiers), nonlogical symbols (individual constants, functions and predicates) which give the language its content, and parentheses to group symbols. Thus, for example, we may use the following symbols:

individual variables: $x, y, z, \cdots$.

logical connectives: $\neg$ (not), $\vee$ (disjunction), $\wedge$ (conjunction), $\rightarrow$ (implication), $\leftrightarrow$ (biconditional).

logical quantifiers: $\forall$ (for all), $\exists$ (there exists).

logical predicate: $=$.

parentheses: $(\,,)$, $[\,,]$.

individual constants: $0, 1, \cdots$.

functions: $+, \cdot, f, g, \cdots$.

predicates: $<$, between, is a natural number, $P, Q, R, \cdots$.

A *term* in the language is defined as follows:

(1) each individual variable and constant is a term; (2) if f is a function of n arguments and $t_1, t_2, \cdots, t_n$ are n terms then $f(t_1, t_2, \cdots, t_n)$ is a term; (3) an expression is a term if and only if it satisfies (1) and (2).

Frequently, functions of two arguments like "+" and "·" are called *operations* and are written between their arguments rather than in front of them.

If P is an n-place predicate and $t_1, t_2, \cdots, t_n$ are n terms then $P(t_1, t_2, \cdots, t_n)$ is called an *atomic formula*. It is customary to write binary predicates between two terms rather than in front of them. So "$x = y$" is an example of an atomic formula and if "B" is the predicate "between" then the expression "$x < y < z$" can be represented by the atomic formula "$B(x, y, z)$".

The *formulas* of the language are just those expressions which are built up from atomic formulas using logical connectives and quantifiers. For example, the following is a formula which expresses the fact that every nonzero element has a reciprocal:

$$(\forall x)\ [x \neq 0 \rightarrow (\exists y)\ (x \cdot y = 1)].$$

(We write "$x \neq 0$" instead of "$\neg (x = 0)$".)

A language is called a *first-order* language if quantification is permitted only on individual variables. The functions and predicates are constants. If there were function or predicate variables and quantification on these variables, the language would be called a *second-order* language.

New symbols can be introduced by definitions. For example, the symbol "2" is an abbreviation for "$1 + 1$", "$x \leqq y$" is an abbreviation for "$x < y \vee x = y$", and subtraction could be defined by the equivalence "$x - y = z \leftrightarrow x = y + z$". Defined terms are always replaceable by undefined terms in all formulas in which they occur. Also, it is convenient to use different types of letters for different types of variables. For example, if "$N(x)$" means "x is a natural number" and if the symbols "n", "m", $\cdots$ are used for variables which range over the set of natural numbers, then "$(\forall n)(\cdots)$" is an abreviation for "$(\forall x)\ (N(x) \rightarrow \cdots)$", and "$(\exists n)\ (\cdots)$" is an abreviation for "$(\exists x)\ (N(x) \wedge \cdots)$".

The *axioms* of our theory are a subset of the set of formulas—intuitively, those formulas which we decide in advance are to be true. For example, the following is one of the Peano axioms in elementary number theory:

$$(\forall n)\ (\forall m)\ (n + 1 = m + 1 \rightarrow n = m).$$

Sometimes a set of axioms is represented by an *axiom schema*—an expression which replaces an infinite number of axioms. An example of such a schema is the principle of mathematical induction: If $\phi(x)$ is any formula then the following is an axiom

$$[\phi(0) \wedge (\forall n)\ (\phi(n) \rightarrow \phi(n + 1)] \rightarrow (\forall n)\ \phi(n).$$

With this example we can see the restrictive nature of a first-order theory. The formula "$\phi(x)$" could be interpreted as "x has the property ϕ" or "x belongs to the set ϕ". (A property is identified with the set of all elements which have the property.) The principle of mathematical induction can then be interpreted as:

If ϕ is a set of natural numbers such that 0 belongs to ϕ and whenever n belongs to ϕ, the successor of n, $n + 1$ belongs to ϕ, then every natural number belongs to ϕ.

But in a first order theory the sets to which the principle of mathematical induction applies are restricted to those sets which are definable by a formula in a first order language (a first order formula). And, as we shall see later, not all sets are definable by such formulas.

Another example is the completeness property of the real numbers—every set of real numbers which is bounded above has a least upper bound. In a first order theory of the real numbers the completeness property only applies to those sets of real numbers which are definable by a first order formula.

Thus, a first order or elementary theory includes a first order language, a set of axioms, and all the first order formulas which can be deduced from the axioms using the rules of logic. A *subtheory* is obtained by omitting some axioms and the subtheory is called *finite* if it only has a finite number of axioms.

A *model* of a first order theory consists of a set of elements (the range of each individual variable) along with interpretations for each of the individual constants, functions and predicates, such that all the axioms of the theory have true interpretations. It should be noted that the interpretation of each individual constant is an element of the set of elements of a model and therefore is included in the range of each individual variable. As an example, let's look at elementary group theory. The nonlogical symbols are e, an individual constant; $\circ$, a function of two arguments; and I, a function of one argument. The axioms are

$$G_1. \quad (\forall x)(\forall y)(\forall z)[(x \circ y) \circ z = x \circ (y \circ z)].$$

$$G_2. \quad (\forall x)(x \circ e = x).$$

$$G_3. \quad (\forall x)(x \circ I(x) = e).$$

G_1 is the associative law for $\circ$, G_2 is the defining property of the identity element e, and G_3 is the defining property of the inverse. ("x^{-1}" is frequently used instead of "$I(x)$".) As a model for this theory we could take the set of natural numbers where "e" is interpreted as "0", "$\circ$" is interpreted as "+", and "I" is interpreted as "$-$", the additive inverse. It is well known that the associative law for addition of natural numbers holds, that $x + 0 = x$ for all natural numbers x, and that $x + (-x) = 0$ for all natural numbers x. Since all the axioms have true interpretations, it follows that we have defined a model of elementary group theory.

At this point the reader should understand the meaning of the compactness theorem; in what follows we give a few of its applications. First, suppose T is a first order theory which has arbitrarily large finite models. For example, the first order theory of fields is such a theory because for each prime p there is a field with p elements. We claim such a theory must have an infinite model. (In particular, there must be a field with an infinite number of elements.) To show that T has an infinite model, for each positive integer n let A_n be the statement that "there exist at least n elements". Thus, for example, A_3 is "$(\exists x)(\exists y)(\exists z)(x \neq y \wedge x \neq z \wedge y \neq z)$". Let T' be the theory obtained from T by adding the infinite set of axioms $A_1, A_2, \cdots$. We write $T' = T + \{A_1, A_2, \cdots\}$. Every finite subtheory of T' has a model, for if the subtheory contains the axioms $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$, $i_1 < i_2 < \cdots < i_k$, then any model of T which has more than i_k elements is a model of the subtheory. Consequently, T' satisfies the hypothesis of the compactness theorem and must therefore have a model. We see that any model of T' is a model of T and also must be infinite.

For our next example, suppose T is the first-order theory of a well-ordering relation. That is, T contains only one nonlogical symbol, a binary predicate, R, and the following axioms:

A_1. $(\forall x)xRx$

A_2. $(\forall x)(\forall y)(xRy \wedge yRx \rightarrow x = y)$

A_3. $(\forall x)(\forall y)(\forall z)(xRy \wedge yRz \rightarrow xRz)$

A_4. $(\forall x)(\forall y)(xRy \vee yRx)$

A_5. If $\phi(x)$ is any formula then the following is an axiom:

$$(\exists x)\phi(x) \rightarrow (\exists x)\,[\phi(x) \wedge (\forall y)\,(\phi(y) \rightarrow xRy)].$$

(R is reflexive, antisymmetric, transitive, and connected, and every nonempty set has an R-smallest element).

One model of T is the set of natural numbers as ordered by $\leqq$. Now let us add to the language of T an infinite number of constants, $a_0, a_1, a_2, \cdots$, and an infinite number of axioms:

B_1. $a_1Ra_0 \wedge a_1 \neq a_0$

B_2. $a_2Ra_1 \wedge a_2 \neq a_1$

. . . . .

B_n $a_nRa_{n-1} \wedge a_n \neq a_{n-1}$

. . . . .

Call the extended system T'. Every finite subtheory of T' has a model because any infinite model of T can be expanded to form a model for any finite subtheory of T'. To see this, suppose the finite subtheory contains the axioms B_2, B_3, B_5, and B_9. (The proof is similar for any other finite set of axioms.) Suppose also that b_0, b_1, b_2, b_3, b_4, b_5, b_8, and b_9 are 8 distinct elements in an infinite model of T such that b_9Rb_8, b_5Rb_4, b_3Rb_2, and b_2Rb_1. (We use the same symbol "R" for both the predicate symbol in the formal language and its interpretation in the model.) Let b_i be the interpretation of a_i, for $i = 1,2,3,4,5,8,9$, and let b_0 be the interpretation of a_i otherwise. Then B_2, B_3, B_5 and B_9 are true and we have a model for this finite subtheory of T'.

Consequently, the compactness theorem implies that T' has a model. Let $b_0, b_1, b_2, \cdots$ be the interpretation of $a_0, a_1, a_2, \cdots$ respectively, in a model of T'. Then we obtain the apparent contradiction that the set $\{b_0, b_1, b_2, \cdots\}$ has no-R-smallest element. That is, we apparently have a model of T' in which one of the axioms of T', namely A_5, is not true. To see what happened let us return to our earlier statement about the restrictive nature of a first order theory. The only possible explanation is that the set $\{b_0, b_1, b_2, \cdots\}$ is not definable by a first-order formula. Thus, it just appears that A_5 is false in the model, while in fact it is not.

One final example. Let T be the first order theory of the real numbers. Add a new individual constant, say ω, to the language of T and add the following axioms to T: $\omega > 1$, $\omega > 2$, $\omega > 3, \cdots$. Then the compactness theorem implies that if T has a model, the extended theory also has a model. The reason for this is that if only a finite number of axioms of the form: $\omega > n_1$, $\omega > n_2, \cdots, \omega > n_k$ are added then in a model of T choose for ω any real number greater than n_1, $n_2, \cdots$, and n_k and it is clear the axioms are satisfied. Therefore, any model of T serves as a model of each finite extension. Thus, non-standard analysis was born.

Since the axioms for the theory of the non-standard reals include all axioms for the standard reals, all theorems and axioms for the standard reals also hold for the non-standard reals. In addition, of course, there is an infinite non-standard real number since ω is greater than every standard natural number. In fact, there are infinitely many infinite numbers because ω, $\omega + 1$, $\omega + 2, \cdots$ are all infinite numbers. Moreover, there must be infinitely small numbers (called *infinitesimals*), because the reciprocal of each number also exists.

Let N be the set of standard natural numbers. N is bounded above by ω, but N has no least upper bound. For any upper bound, b, of N must be infinite, $b - 1 < b$ and $b - 1$ is also an upper bound of N. It appears that the non-standard reals do not satisfy the completeness property. But, of course, this is not the case. The completeness property only applies to sets of real numbers definable by a first order formula. Consequently, the set of standard natural numbers is not definable by a first order formula. Similarly we see there are many other sets not definable by a first order formula, for example the set of standard real numbers, the set of standard rational numbers and the set of infinitesimals.

Perhaps the preceding examples give some idea of the strength and limitations of the compactness theorem and indicate some of its applications in logic.

(I should like to thank W. R. Fuller for many suggestions and comments which helped me to prepare this paper, and I'd also like to thank my typist, Elizabeth Young, who patiently and accurately typed several drafts.)

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COVERING SETS OF CONGRUENCES

MARTIN BILLIK and HUGH M. EDGAR, California State University, San Jose

1. Introduction. In a paper in 1952, Erdös [1] introduced the following concept:

DEFINITION 1. *Let $1 < n_1 < n_2 < \cdots < n_k$, each $n_i \in Z$ (the rational integers). Let $0 \leqq b_i < n_i$ for each value of i. The set of ordered pairs $\{(b_1, n_1), \cdots, (b_k, n_k)\}$ of* model, the extended theory also has a model. The reason for this is that if only a finite number of axioms of the form: $\omega > n_1$, $\omega > n_2, \cdots, \omega > n_k$ are added then in a model of T choose for ω any real number greater than n_1, $n_2, \cdots$, and n_k and it is clear the axioms are satisfied. Therefore, any model of T serves as a model of each finite extension. Thus, non-standard analysis was born.

Since the axioms for the theory of the non-standard reals include all axioms for the standard reals, all theorems and axioms for the standard reals also hold for the non-standard reals. In addition, of course, there is an infinite non-standard real number since ω is greater than every standard natural number. In fact, there are infinitely many infinite numbers because $\omega, \omega + 1, \omega + 2, \cdots$ are all infinite numbers. Moreover, there must be infinitely small numbers (called *infinitesimals*), because the reciprocal of each number also exists.

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Erdös made two conjectures concerning covering sets of congruences which remain unresolved:

CONJECTURE 1. *There is no covering set of congruences in which all the moduli are odd.*

CONJECTURE 2. *For each positive integer N there is a covering set of congruences* $\{(b_1, n_1), \cdots (b_k, n_k)\}$ *in which* $n_1 > N$.

Contributions to the problem have been made by Mirsky and Newman [1], Davenport and Rado [1], Swift [2], Selfridge [3], Stein [4], Curzio [5], Jordan [6], Szemerédi [7], Churchhouse [8], Crittenden and Vanden Eynden [9], Krukenberg [10], Dewar [11], and others. We wish to give some additional partial results on several facets of the problem and pose some new questions.

2. A new formulation of the problem. We first make the following definition:

DEFINITION 2. *A covering set of congruences will be said to be a minimal covering set of congruences (MC) if deletion of any ordered pair* (b_i, n_i) *results in the remaining set of ordered pairs* $\{(b_1, n_1), \cdots (b_{i-1}, n_{i-1}), (b_{i+1}, n_{i+1}), \cdots (b_k, n_k)\}$ *failing to constitute a covering set of congruences.*

We shall assume henceforth that $\{(b_1, n_1), \cdots (b_k, n_k)\}$ is an MC. Consider the ring $Z/_{n_1 Z} \times Z/_{n_2 Z} \times \cdots \times Z/_{n_k Z}$ where addition and multiplication are defined coordinatewise and are reckoned modulo n_i in the ith coordinate. Define the mapping $h: Z \to Z/_{n_1 Z} \times \cdots \times Z/_{n_k Z}$ by $z \mapsto h(z) = (c_1, c_2, \cdots c_k)$ provided $z \equiv c_i (\text{mod}\, n_i)$ for each i. Then h is a ring homomorphism. If $\{(b_1, n_1), \cdots (b_k, n_k)\}$ is our MC, then it is immediate that $c_i = b_i$ must occur for at least one value of i, regardless of the choice of the integer z. Also for each choice of i there exists $z \in Z$ with $h(z)$ and $(b_1, \cdots b_k)$ agreeing only in the ith coordinate. If $H = h(Z)$ then $H \cong Z/\ker h = Z/LZ$ where $L = [n_1, n_2, \cdots n_k]$, the least common multiple of the moduli. Hence H is a cyclic group of order L. It can be shown that each element of $h(Z)$ must differ from $(b_1, \cdots b_k)$ in at least three places (three being the best possible absolute constant). This result can be rephrased to say that no $(k-2)$ of the k congruences can be solved simultaneously.

3. Divisibility conditions. We first prove that each modulus must divide the least common multiple of the other $(k-1)$ moduli. Suppose, for instance, that $n_1 \nmid [n_2, \cdots, n_k]$. Then $h([n_2, \cdots, n_k]) = (c_1, 0, \cdots 0)$ where $c_1 \neq 0$. From Section 2 we know that there is a k-tuple of the form $(b_1, x, x, \cdots x)$ in $h(Z)$, where an "x" in the ith coordinate indicates that the entry in the ith coordinate isn't b_i. Hence $(b_1 + c_1, x, x, \cdots, x) \in h(Z)$, and so, again by Section 2, we must have $b_1 + c_1 \equiv b_1$ $(\text{mod}\, n_1)$, i.e., $n_1 \mid c_1$ but this is a contradiction.

Next we prove the following:

THEOREM. *Let p be a prime with $p^c \| n_i$ for some value of i while $p^{c+1} \nmid n_j$ obtains for all values of j. Then $p^c \| n_j$ for at least p values of j.*

Proof. Suppose, upon relabelling the moduli if necessary, that $p^c \| n_1$, $p^c \| n_2, \cdots, p^c \| n_s$ while $p^c \nmid n_j$ whenever $s < j \leqq k$. We need to prove that $s \geqq p$. Put $L = [n_{s+1}, \cdots n_k]$ and $p^d \| L$ where $d < c$ holds. Let $(b_1, x_2, x_3, \cdots x_k)$ denote an element of $h(Z)$ in which the ith coordinate, $2 \leqq i \leqq k$, is $x_i \neq b_i$. Since $(L, L, \cdots L, 0, 0, \cdots 0)$ and $(b_1, x_2, x_3, \cdots x_k) \in h(Z)$ at least one of the following congruences must hold: $b_1 + L \equiv b_1 \pmod{n_1}$, $x_2 + L \equiv b_2 \pmod{n_2} \cdots x_s + L \equiv b_s \pmod{n_s}$. Using

$$(2L, 2L, \cdots 2L, 0, \cdots 0),\ (3L, 3L, 0, \cdots 0), \cdots ((p-1)L, (p-1)L, \cdots (p-1)L, 0, \cdots 0)$$

in the same way we obtain the following $(p-1) \times s$ matrix of congruences:

$$b_1 + L \equiv b_1 (\bmod n_1),\ x_2 + L \equiv b_2 (\bmod n_2), \cdots, x_s + L \equiv b_s (\bmod n_s).$$

$$b_1 + 2L \equiv b_1 (\bmod n_1),\ x_2 + 2L \equiv b_2 (\bmod n_2), \cdots, x_s + 2L \equiv b_s (\bmod n_s).$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$b_1 + (p-1)L \equiv b_1 (\bmod n_1),\ x_2 + (p-1)L \equiv b_2 (\bmod n_2), \cdots, x_s + (p-1)L = b_s (\bmod n_s).$$

No one of the congruences in the first column holds so we obtain a $(p-1) \times (s-1)$ matrix of congruences upon deleting the first column of congruences. In each row of the new matrix of congruences at least one congruence must hold. If $s < p$ it follows by the box principle that there must be two congruences satisfied in some column but this is easily seen to be impossible.

COROLLARY 1. *Each modulus must divide the least common multiple of the other $(k-1)$ moduli.*

COROLLARY 2. *If a prime p is a factor of one of the moduli then it is necessarily a factor of at least p of the moduli.*

4. Antithesis of the Chinese remainder theorem. If $\{(b_1, n_1), \cdots (b_k, n_k)\}$ is to constitute an MC an obvious necessary condition on the moduli is $\sum_{i=1}^{k} 1/n_i \geqq 1$. Mirsky and Newman [1], Davenport and Rado [1] and Stein [4] have shown $\sum_{i=1}^{k} 1/n_i > 1$ to be a necessary condition. These proofs are all nonelementary in the technical sense and no elementary proof is known. Recall the following definitions: $\tau(n)$ is the number of positive integral divisors of the positive integer n, $\sigma(n)$ is the sum of the positive integral divisors of n, n is abundant if and only if $\sigma(n) > 2n$ and finally n is perfect if and only if $\sigma(n) = 2n$. With $L = [n_1, n_2, \dots n_k]$, since $\sigma(L) - L \geqq \sum_{i=1}^{k} L/n_i$ we see that the abundance of L is a consequence of $\sum_{i=1}^{k} 1/n_i > 1$. We give an elementary proof of the necessity of the condition $\sigma(L) > 2L$ in case L is even. $\sigma(L) \geqq 2L$ is a consequence of the elementary result $\sum_{i=1}^{k} 1/n \geqq 1$ and the strict inequality obtains unless L is a perfect number and n_i runs over all divisors of L except the unit divisor. It is not known whether there are any odd perfect numbers and (Conjecture 1, Section 1) it is not known whether there are any minimal covers with L odd so we confine ourselves to the case of even L. If L is an even perfect number then it is well known that $L=2^{p-1}(2^p-1)$ where 2^p-1 is a Mersenne prime. We can always claim (Corollary 2, Section 3) that $P_L \leqq k \leqq \tau(L)-1$, where P_L is the largest prime divisor of L and this leads to the contradiction $(2^p-1) \leqq k \leqq (2p-1)$ in the current context. (The case $L=6$ doesn't arise because $L \geqq 12$ always obtains.)

We add one more technically nonelementary proof of $\sum_{i=1}^k 1/n_i > 1$ to the list indicated above. To assume that $\sum_{i=1}^k 1/n_i = 1$ is to assume that for every r such that $0 \leqq r < L$ there is exactly one value of i for which $r \equiv b_i (\text{mod } n_i)$ obtains. To the ordered pair (b_i, n_i) we make correspond the polynomial $f_i(x) = (x\zeta^{-b_i})^{L/n_i} - 1$, where ζ is a primitive Lth root of unity. The assumption $\sum_{i=1}^k 1/n_i = 1$ now takes the form $\prod_{i=1}^k f_i(x) = c(x^L - 1) = c\prod_{r=0}^{L-1}(x-\zeta^r)$ for some appropriate complex constant c since $f_i(x)$ has degree L/n_i and has ζ^r as its roots, r as above. However, the coefficient of

$$x^{L/n_1 + L/n_2 + \cdots + L/n_{k-1}}$$

is

$$-\zeta^{-L(b_1/n_1 + b_2/n_2 + \cdots + b_{k-1}/n_{k-1})} \neq 0$$

which is a contradiction.

We prove the following:

THEOREM (Antithesis of the Chinese remainder theorem). *Let* $1 < n_1 < n_2 < \cdots < n_k$, *each* $n_i \in Z$ (*the rational integers*). *Let* $0 \leqq b_i < n_i$ *for each value of* i. *Assume that* $(n_i, n_j) \nmid (b_i - b_j)$ *obtains for all* $i \neq j$, $1 \leqq i, j \leqq k$. *Then there exists a* $z \in Z$ *satisfying* $z \not\equiv b_i (\text{mod } n_i)$ *for all* i *with* $1 \leqq i \leqq k$.

Proof. Suppose the contrary. Then the set of ordered pairs $\{(b_1, n_1), \ldots (b_k, n_k)\}$ forms a covering set of congruences so that we must have $\sum_{i=1}^k 1/n_i > 1$. However, this inequality is equivalent to the statement that at least two of the congruences are simultaneously solvable and this statement, in turn, is equivalent to saying that $(n_i, n_j) \mid (b_i - b_j)$ holds for at least one pair (i,j) with $i \neq j$, $1 \leqq i, j \leqq k$.

5. Implications of all odd moduli. We list some consequences of the assumption that L is odd:

(1) If Conjecture 2 is false, then any covering set with maximal n_1 will necessarily have L even.

(2) If the canonical decomposition of L is written $L = p_1^{e_1} \cdots p_r^{e_r}$ then we must have $r \geqq 5$. If, in addition, L is squarefree then we must have $r \geqq 6$. (These results follow essentially from the work of Churchhouse [8]).

(3) Define the map $f: H \mapsto H$ by $h \mapsto 2h$ for every $h \in H$, H as in Section 2. Then f is a fixed-point-free outer automorphism of H. If we denote by s the order of the element f in the group Aut H so that $2^s \equiv 1 \pmod L$ with s minimal then

$$\log_2 L < s \mid \lambda(L) \mid \frac{\phi(L)}{2^{r-1}}$$

where $\lambda(L)$ is the so-called universal exponent of L (see [12], p. 53). If we assume that $s = q =$ prime then we must have $p_i \equiv 1 (\text{mod}\, q)$ for every i such that $1 \leqq i \leqq r$ and $L > (2^{20} \cdot 5)^5$ must occur.

(4) Each modulus must be a divisor of the least common multiple of any $(k-2)$ of the remaining moduli.

Proof of (1). Suppose $\{(b_1, n_1), (b_2, n_2), \dots (b_k, n_k)\}$ is a covering set of congruences in which $n_1 \equiv n_2 \equiv \cdots \equiv n_k \equiv 1 \pmod 2$. Then $\{(b_1 + n_1, 2n_1), (b_1 + 2n_1, 4n_1), (b_1 + 4n_1, 6n_1), (b_1 + 4n_1, 8n_1), (b_1 + 8n_1, 12n_1), (b_1, 24n_1), (b_2, n_2), \cdots (b_k, n_k)\}$ is also a covering set of congruences because $\{(1,2), (2,4), (4,6), (4,8), (8,12) (0,24)\}$. is

Proof of (3) (Sketch). The map is surjective because H is of order L and L is assumed to be odd.

$$\lambda(L) = \phi(L) \Bigg/ \left(\frac{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)}{[p_1 - 1, p_2 - 1, \cdots, p_r - 1]} \right).$$

Since $2^{d_1 + d_2 + \cdots + d_r} \,\|\, (p_1 - 1)(p_2 - 1) \cdots (p_r - 1)$ while $2^{\max\{d_1, d_2 \cdots d_r\}} \,\|\, [p_1 - 1, p_2 - 1, \cdots, p_r - 1]$ it follows that $\lambda(L) \mid \phi(L)/2^{r-1}$. If $s = q =$ prime then it is known (see [13], p. 19) that every factor of $M_q = 2^q - 1$ is of the form $2mq + 1$ for suitable nonnegative integral m. Since $2^q \equiv 1 \pmod L$ we must have $2^q \equiv 1 \pmod{p_i^{e_i}}$ for every i such that $1 \leqq i \leqq r$. It follows that $q \mid \phi(p_i^{e_i}) = p_i^{e_i - 1}(p_i - 1)$ for all such i. Now if $q \mid p_i$ for some i then $q \mid L \mid 2^q - 1 = M_q$ which is known to be impossible. Hence $q \mid (p_i - 1)$ must obtain for all values of i under discussion. Since L must have, by (2), at least 5 distinct prime factors so also must M_q and thus (see [14], p. 70) $q \geqq 83$. If we assume that L has five distinct prime divisors p_1, p_2, p_3, p_4, and p_5, then $p_5 \geqq 14q + 1$ because of the requirements $q \mid (p_i - 1)$ for every i (the others would be $p_1 = 2q + 1$, $p_2 = 6q + 1$, $p_3 = 8q + 1$ and $p_4 = 12q + 1$ with $q \equiv 2 \pmod 3$). With $q \geqq 83$, Corollary 2 of Section 3 implies that L must have at least $14 \cdot 83 + 1 = 1163$ divisors from which it follows that $L > (2^{20} \cdot 5)^5$.

6. A further problem. If $g = (n_1, n_2, \cdots n_k)$ then we have seen that $g = 1$ and $g = 2$ can occur. Recently Dewar [11] has shown that $g = 3$ and $g = 4$ can both occur. What other values, if any, can g adopt?

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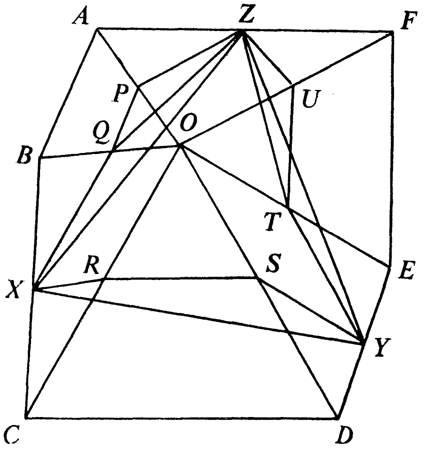
THE ASYMMETRIC PROPELLER

LEON BANKOFF, Los Angeles, California, PAUL ERDÖS, University of Wisconsin, Madison, and MURRAY S. KLAMKIN, Ford Motor Company, Dearborn, Michigan

In this note, we prove an extension of a known elementary geometric result in two ways, i.e., synthetically and by complex numbers. Then we show that the result characterizes closed curves of 6-fold symmetry.

THEOREM. *If OAB, OCD, OEF are equilateral triangles, each labeled in the same clockwise or counterclockwise direction (and not necessarily congruent), then X, Y, Z, the midpoints of BC, DE and FA, are vertices of an equilateral triangle.*



Synthetic Proof. Let P, Q, R, S, T, U denote the midpoints of OA, OB, OC, OD, OE, and OF, respectively. Then,

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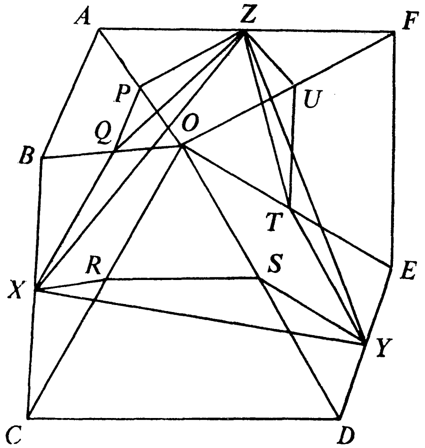


Synthetic Proof. Let P, Q, R, S, T, U denote the midpoints of OA, OB, OC, OD, OE, and OF, respectively. Then,

$$PQ = PO = ZU \quad \text{and} \quad \measuredangle\,(PQ, ZU) = 60°;$$

$$PZ = OU = UT \quad \text{and} \quad \measuredangle\,(PZ, UT) = 60°.$$

Thus, $\measuredangle QPZ = \measuredangle ZUT$ and triangles QPZ and ZUT are congruent with a 60° mutual inclination between corresponding sides. Then, $QZ = ZT$ with $\measuredangle QZT = 60°$. Since $QX = OR = OS = TY$ with $\measuredangle\,(QX, TY) = 60°$, triangles ZQX and ZTY are congruent. Finally, $ZX = ZY$ with $\measuredangle XZY = 60°$, giving the desired result.

NOTE. Triangles ZQT, YUR and XSP are equilateral. This can be shown directly, as with triangle ZQT above, or by allowing one of the triangles OAB, OCD, OEF to degenerate to a point-triangle at O and applying the main theorem.

This proof applies for any rotation of one or more of the triangles OAB, OCD and OEF about O. Thus the triangles in the initial configuration may be separate, contiguous or overlapping in any manner.

Proof by Complex Numbers. In the figure, $z_1(OA)$, $z_2(OC)$, $z_3(OE)$ denote arbitrary complex numbers and $\lambda = e^{i\pi/3}$. We now have to show that $\lambda z_1 + z_2$, $\lambda z_2 + z_3$, $\lambda z_3 + z_1$ are the vertices of an equilateral triangle, i.e.,

$$(\lambda z_3 + z_1) - (\lambda z_2 + z_3) = \lambda^2\{(\lambda z_2 + z_3) - (\lambda z_1 + z_2)\}$$

or

$$\lambda(z_3 - z_2) + z_1 - z_3 = \lambda^3(z_2 - z_1) + \lambda^2(z_3 - z_2).$$

Since $\lambda^3 = -1$ and $\lambda - \lambda^2 = 1$, the result follows.

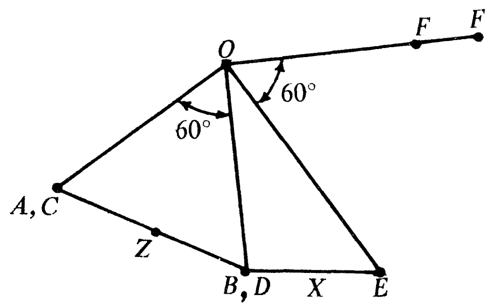
Proofs using complex numbers may also be found in the Amer. Math. Monthly, Aug.–Sept. 1968, Problem B–1 of the William Lowell Putnam Mathematical Competition and in H. Eves, *A Survey of Geometry*, II, Allyn and Bacon, Boston, 1965, p. 184. In these two solutions, however, the superfluous conditions $|z_1| = |z_2| = |z_3|$ as well as non-overlapping were assumed.

We now show that the result given by our theorem characterizes curves of 6-fold symmetry.

THEOREM. *AB, CD, EF are arbitrary chords (in the same sense) of a given closed curve, starlike with respect to O, and which subtend 60° angles from point O. If X, Y, Z, the respective midpoints of DE, FA, BC, are vertices of an equilateral triangle, then the curve must be one of 6-fold symmetry (with respect to O).*

Proof. We first show that there exists a chord PQ such that POQ is equilateral. Let OR be a shortest radius from O to the curve and then let OR' denote the radius making 60° with OR. It follows by continuity that as we rotate the radius OR about O up to 60°, $OR' - OR$ must have a zero value. An equilateral triangle POQ still exists even if we dropped the starlike assumption for the curve. In this case, we would apply P. Lévy's chord theorem (see H. Hadwiger, H. Debrunner, V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, N.Y., 1964, p. 23).

Let points A and C be fixed points coinciding with P and let points B and D be fixed coinciding with Q. OE is an arbitrary radius and $OF' = OE$.



It follows from our first theorem that the midpoint of AF must coincide with that of AF'. Thus, $OF = OF'$ and the curve is one of 6-fold symmetry.

WALKING IN THE RAIN, RECONSIDERED

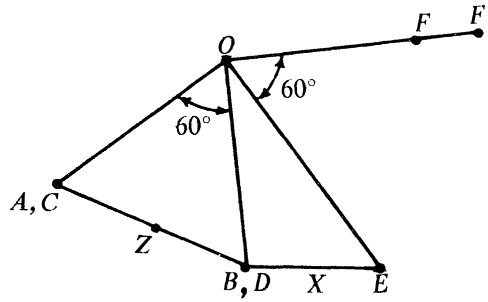
B. L. SCHWARTZ, McLean, Virginia and
M. A. B. DEAKIN, Monash University, Clayton, Australia.

Introduction. In an earlier paper [1], one of the authors studied the best strategy according to which a man might walk (or run) in the rain from point A to point B in order to minimize the amount of wetting he undergoes. This paper corrects a technical error in the earlier work and, in doing so, simplifies and extends the earlier results. Notation and terminology follow the earlier paper. However, for completeness, we restate the definitions needed to make this paper self-contained.

The model. Let $\boldsymbol{i}$ be a unit vector in the direction $\overrightarrow{AB}$; $\boldsymbol{k}$ a unit vector pointing upwards; and $\boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{i}$. The rain, assumed a fluid of uniform density, is falling at speed V_T, and being swept along by a horizontal wind of velocity $V_T(w\boldsymbol{i} + W\boldsymbol{j})$. Thus the velocity of the rain is $V_T(w\boldsymbol{i} + W\boldsymbol{j} - \boldsymbol{k})$, as seen from a fixed point on the ground. The man's velocity is $\boldsymbol{i}\, V_T x$, where x is to be determined. Thus, the relative velocity of the rain droplets, as seen by the man is $V_T\{(w-x)\boldsymbol{i} + W\boldsymbol{j} - \boldsymbol{k}\}$. The man is modelled as a cuboid, getting wet on three of his six sides: front or back, right or left, and top. These are taken to have areas A, ηA, and εA respectively.

Wetness function. The amount of rain impinging on the top surface per unit time is proportional to the area εA multiplied by the velocity component of the rain normal to the surface. Thus the amount of rain falling on the man's head is proportional to εA. Similarly, the amount striking the side surface is proportional to $\eta A\,|W|$; and finally the amount impinging on front or back is proportional to $A\,|w - x|$.

Let points A and C be fixed points coinciding with P and let points B and D be fixed coinciding with Q. OE is an arbitrary radius and $OF' = OE$.



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The absolute values in the last two expressions arise from the observation that just two of the four vertical sides can be wetted by rain, depending on the signs of W and $(x - w)$.

Thus the total amount of rain falling on the man per unit time is proportional to $\varepsilon + \eta|W| + |w - x| = \phi + |w - x|$, where ϕ is defined as $\varepsilon + \eta|W|$.

The total amount falling on the man in his trip from A to B is directly proportional to the amount landing on him per unit time, and inversely proportional to his speed—i.e., it is proportional to

$$F(x) = \frac{\phi + |w - x|}{x} \tag{1}$$

which we refer to as the "wetness function".

This differs in form from the corresponding expression in the earlier paper. The earlier paper erred in the inclusion of a factor $\sqrt{(w-x)^2 + W^2 + 1}$ in the denominator.

Heuristic confirmation. The previous (incorrect) form for $F(x)$ had the property that it became arbitrarily small for sufficiently large x. This implied that the man could reduce his soaking to as small a level as he wished if he could attain a great enough speed. Clearly, this is not true. If he goes extremely fast from A to B, the man sweeps out, as an irreducible minimum, all the rain that is in the prism whose long axis is the line segment AB, and whose cross section is the same as the man's own. This quantity of rain is a positive constant; it cannot be made arbitrarily small.

The corrected version of $F(x)$ given above does have a positive limiting value (unity) approached when x becomes large, and hence passes this intuitive test.

Optimal policy. We seek to minimize the function $F(x)$. Observe that $F(\cdot)$ has a discontinuous derivative at $x = w$.

$$F'(x) = (\phi + w)/x^2 \text{ for } x < w \tag{2}$$

and

$$F'(x) = -(\phi - w)/x^2 \text{ for } x > w. \tag{3}$$

Thus we cannot minimize $F(\cdot)$ by cookbook calculus. The shape of the graph of $F(x)$ depends on the relative values of ϕ and w. If $\phi > w$, then $F(x)$ is monotone decreasing for all x (see Figure 1). Hence the optimal strategy for the man is to run as fast as he can.

On the other hand, if $\varphi \leqq w$, then $F(x)$ has a minimum at $x = w$ (see Figure 2). In this case, the man should travel at the speed $x = w$, if he can, so as to avoid encountering moisture on front or back. He will get wet only on the top and side. If he cannot attain w, the speed of the wind, then he is operating in the portion of the $F(x)$ curve that is monotone decreasing, and hence again should run as fast as he can.

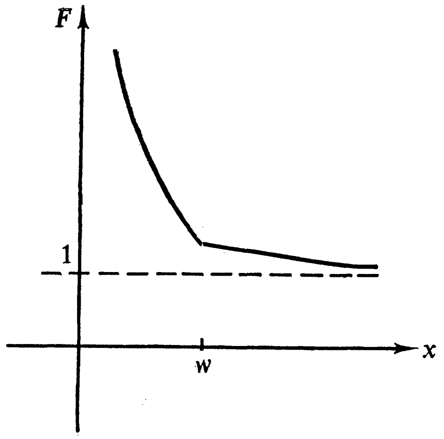

FIG. 1

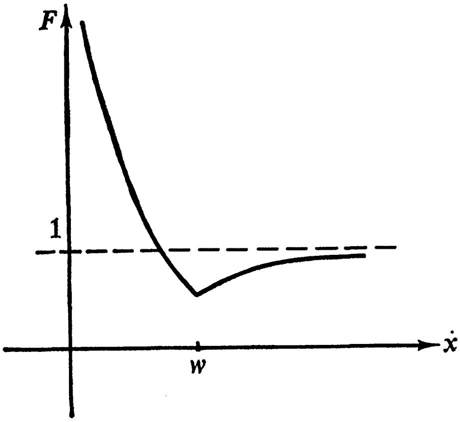

FIG. 2

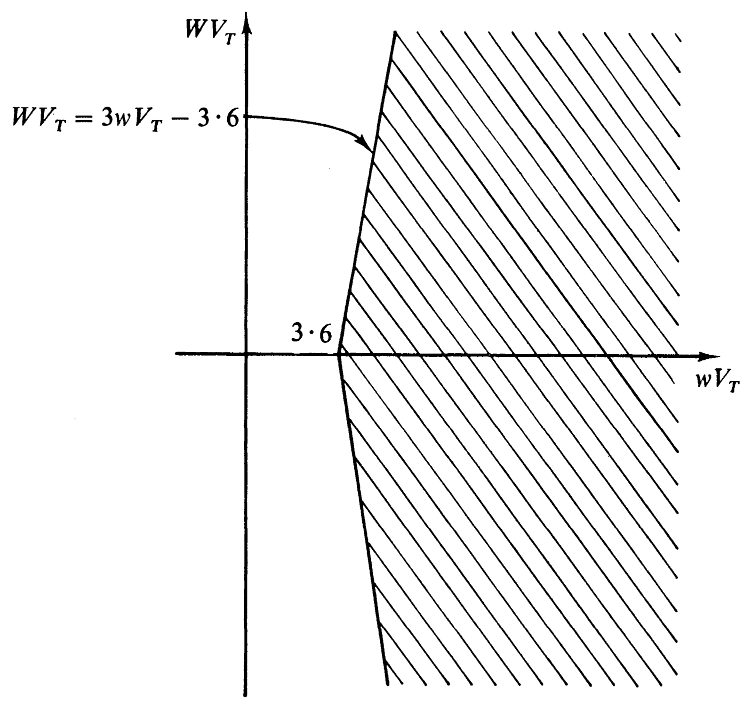

FIG. 3

Decision procedure. The decision as to whether or not to run at top speed is dependent upon whether $\phi > w$. It is best to match one's speed to that of the relevant component of the wind if $\phi < w$, i.e., if

$$\varepsilon + \eta |W| < w. \tag{4}$$

Inequality (4) defines a region of the (w, W) plane for which this policy is optimal. This is graphed in terms of the "real" variables WV_T, wV_T in Figure 3, the values of ε, η, V_T being as presented in the earlier analysis: $\varepsilon = 0.06$, $\eta = 0.33$, $V_T = 20$ m.p.h.

For this case, inequality (4) becomes

$$|WV_T| < 3wV_T - 3.6\,. \tag{5}$$

The decision rule is simply this: If the terminal point of the vector $V_T(w, W)$ lies in the shaded area of Figure 3, the man should proceed at speed w, if possible. In all other cases, he should run as fast as he can.

Sensitivity analysis. Suppose $\phi < w$, but that the man still runs as fast as he can. The extent to which he gets wet is measured by the wetness function $F(X)$ where X is his top speed (in terms of the terminal velocity of falling raindrops). Define

$$R = \frac{F(X)}{F(w)}\,. \tag{6}$$

R will now measure the penalty incurred in following the naive rather than the more sophisticated strategy. We consider only the case $X > w$, as otherwise the question of the two strategies does not arise.

Substitution from (1) into (6) gives

$$R = \frac{\phi + X - w}{X} \cdot \frac{w}{\phi}$$

from which it is clear that R becomes larger as ϕ is decreased. The largest possible value of R will thus occur when ϕ achieves its minimum value ε, i.e., when $W = 0$. Under these circumstances

$$R = \frac{w}{X}\left(1 + \frac{X - w}{\varepsilon}\right).$$

R is maximized when

$$X = 2w - \varepsilon$$

and achieves the value

$$R_{\max} = \frac{(X + \varepsilon)^2}{4X\varepsilon}\,.$$

For the case studied in the earlier paper: $\varepsilon = 0.06$, $X = 1$, this assumes the value 4.68, so that under quite feasible conditions a runner following the naive strategy gets more than four times as wet as a jogger matching his speed to that of the rain.

Conclusion. We have shown that the correction of a technical error in the earlier analysis [1] simplifies the analysis without seriously altering the conclusions. The amended analysis provides a reasonably accessible and plausibly motivated case of a minimum of the $|x|$ type and may be of some use on this account to teachers of elementary college calculus. The simplifications inherent in this paper also suggest the feasibility of more realistic models; e.g., it may be possible to relax the assumptions that the rain is all falling in exactly the same direction or with the same speed. We have not, however, attempted any such generalizations ourselves.

Reference

1. M. A. B. Deakin, Walking in the rain, this MAGAZINE, 45 (1972) 246–253.

AN "ELEPHANTINE" EQUATION

S. NARANAN, Tata Institute of Fundamental Research, Bombay

There are many legendary tales about the wit and wisdom of Birbal, poet, musician, intellectual and the favorite courtier of the Indian Emperor Akbar (1556–1605). Once, three men came to Akbar's court with the problem of sharing the wealth left by their deceased father. The old man had willed that the eldest son should get 1/2, the second son 1/3, and the youngest son 1/9 of his entire property.

The entire property consisted of 17 elephants. It was even suggested that some of the elephants might be slaughtered, since—as the saying goes—"an elephant is worth a thousand gold coins, alive or dead." Birbal's solution was neat and clean. He ordered the Royal Elephant to be lined up with the 17 elephants. The men took their shares of 9, 6, and 2 elephants, leaving behind the Royal Elephant!

Birbal's solution exploited the fact that $1/2+1/3+1/9=17/18$ (not 1). Are there any triplets x_1, x_2, x_3 besides 2,3,9 which have a similar property? In other words, what are the solutions of the Diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=\frac{E}{E+1} \tag{1}$$

where

(a) x_1, x_2, and x_3 are unequal, and

(b) $E+1$ is the least common multiple of x_1, x_2, x_3?

It turns out that there are 7 solutions. One of the x's must be 2. Otherwise, the maximum value of the left-hand side of (1) is $1/3+1/4+1/5=47/60$; since the least common multiple of the x's must be at least 12, the right-hand side is at least $11/12$. But $47/60$ is already less than $11/12$. Therefore there is no solution without 2 as one of the x's. Arguing similarly, it can be shown that another of the x's must be 3 or 4. Taking $x_1=2$ and $x_2=3$, x_3 may be relatively prime to 2 and 3 or of the form $2m$, $3m$, or $6m$. Substituting in (1), the corresponding solutions for x_3 are 7, 8, 9, and 12. Taking $x_1=2$ and $x_2=4$, x_3 may be relatively prime to 2 and 4 or of the form of a minimum of the $|x|$ type and may be of some use on this account to teachers of elementary college calculus. The simplifications inherent in this paper also suggest the feasibility of more realistic models; e.g., it may be possible to relax the assumptions that the rain is all falling in exactly the same direction or with the same speed. We have not, however, attempted any such generalizations ourselves.

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$2m$ or $4m$. This gives $x_3 = 5, 6, 8$. The values of E in the seven solutions are, respectively, 41, 23, 17, 11, 19, 11, and 7.

We may generalize the problem to n partitions; that is, to find solutions in positive integers of

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = \frac{E}{E+1}, \tag{2}$$

where E is the least common multiple of $x_1, x_2, \cdots, x_n$. It is readily seen that

$$E = 2^n - 1,\ x_i = 2^i,\ i = 1, 2, \cdots, n$$

is a solution for any positive integer n, since

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

There are many solutions for the general case [1, pp. 688–91]. It is interesting to consider the *minimal solution* E_n for a given n. For $n = 2, 3, 4$, $E_n = 2^n - 1$. But for $n \geqq 5$, $2^n - 1$ is not the minimal solution. For instance, for $n = 5$, $E_n = 23$ (not 31):

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{12} = \frac{23}{24}.$$

n	E_n	Number of solutions
2	3	1
3	7	1
4	15	1
5	23	1
6	35	1
7	59	8
8	59	1
9	89	1
10	119	3
11	119	2
12	179	13
13	179	1
14	239	9
15	239	1

TABLE 1.

Table 1 lists the minimal solutions for n up to 15 and the number of n-tuples $(x_1, x_2, \cdots, x_n)$ which yield the minimal value of E.

How are the minimal solutions obtained? We start by finding the smallest number that has a given number D of divisors (including 1 and the number itself). If $D = abc\cdots$, then it is well known that the numbers with D divisors are of the form $p^{a-1}q^{b-1}r^{c-1}\cdots$, $p, q, r, \cdots$ being prime numbers. The smallest number with D

divisors is found by suitably choosing $a, b, c, \cdots$ and $p, q, r, \cdots$. For instance, for $D = 16$, 16 can be written as $2\cdot 2\cdot 2\cdot 2$ or $4\cdot 2\cdot 2$ or $4\cdot 4$ or $8\cdot 2$ or 16. Therefore all numbers with 16 divisors are of the form $pqrs$ or p^3qr or p^3q^3 or p^7q or p^{15}.It turns out that $2^3\cdot 3\cdot 5 = 120$ is the smallest number with 16 divisors. Then we choose out of the D divisors, the maximum number n that would satisfy (2). For instance, for $D = 16$, $E + 1 = 120$, and its 16 divisors are

$$1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$$

and the sum of their reciprocals is 240/120. To get 119 in the numerator instead of 240, we remove the terms 60, 40, 20, 1 or 60, 40, 15, 6. We have thus two solutions with $n = 11$, $E = 119$:

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{24}+\frac{1}{30}+\frac{1}{40}+\frac{1}{60}+\frac{1}{120}=\frac{119}{120}$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{20}+\frac{1}{24}+\frac{1}{30}+\frac{1}{40}+\frac{1}{60}=\frac{119}{120}.$$

The author is grateful to the referee for valuable suggestions.

Reference

1. L. E. Dickson, History of the Theory of Numbers, vol. II, Chelsea, New York, 1952.

ANOTHER SIMPLE SOLUTION OF THE BUTTERFLY PROBLEM

STEVEN R. CONRAD, B. N. Cardozo H. S., Bayside, N. Y.

As far as this author knows, the Butterfly Problem has a history dating back to 1815 [7], and the problem has appeared sporadically in the literature since that time [1–17]. The problem is usually presented [2] as follows: Through M, the midpoint of chord AB of a circle, two other chords, CD and EF, are drawn. ED intersects AB in P and CF intersects AB in Q. Prove that $PM = QM$ (Figure 1).

Proofs of this "butterfly property" run the gamut from the elementary synthetic proofs [1, 6, 7, 9, 11, 12, 13, 15] to the shorter projective proofs [1, 5, 8, 10, 14, 16]. Generalizations have been offered to show that the property is true for any conic [1, 2, 3]. In addition, many related properties have been offered [4, 6, 16, 17]. The property has even been elevated to the status of a listed theorem in at least one textbook [14]. The statement of M. S. Klamkin [4] that "An elementary synthetic proof of this result is somewhat involved" is somewhat exaggerated, as is evidenced in [1]. However, it is true that the proofs found in print do generally require the introduction of some auxiliary lines. Further, many of these proofs, though short and elegant, would appear somewhat sophisticated to the beginning student. What is needed is divisors is found by suitably choosing $a,b,c,\cdots$ and $p,q,r,\cdots$. For instance, for $D=16$, 16 can be written as $2\cdot 2\cdot 2\cdot 2$ or $4\cdot 2\cdot 2$ or $4\cdot 4$ or $8\cdot 2$ or 16. Therefore all numbers with 16 divisors are of the form $pqrs$ or p^3qr or p^3q^3 or p^7q or p^{15}. It turns out that $2^3\cdot 3\cdot 5=120$ is the smallest number with 16 divisors. Then we choose out of the D divisors, the maximum number n that would satisfy (2). For instance, for $D=16$, $E+1=120$, and its 16 divisors are

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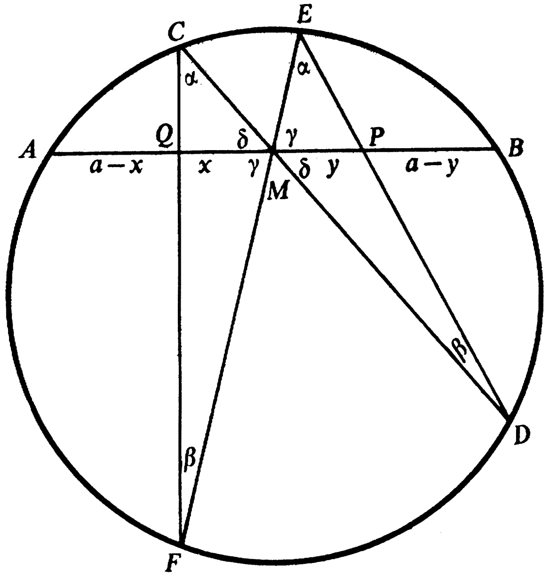

FIG. 1.

In order to simplify the notation of the proof, several preliminary items will be noted below and in Figure 1. There are many pairs of equal angles in Figure 1. Several of these pairs are worth identifying by Greek letter names, as follows:

Let equal angles FCD and DEF be called α.

Let equal angles CFE and EDC be called β.

Let equal angles QMF and EMP be called γ.

Let equal angles CMQ and DMP be called δ.

In addition, let $QM = x$, $PM = y$, and let equal segments AM and MB each be called a. Then, $AQ = a - x$ and $BP = a - y$.

One more notational convenience will be employed. The symbol $K(RST)$ will denote the area of triangle RST. With the notation thus simplified, the elegance of the proof will be much more apparent.

From Figure 1, we see that,

$$\frac{K(QCM)}{K(PEM)} \cdot \frac{K(PEM)}{K(QFM)} \cdot \frac{K(QFM)}{K(PDM)} \cdot \frac{K(PDM)}{K(QCM)} = 1.$$

Hence, $$\frac{CM \cdot CQ \cdot \sin\alpha}{EM \cdot EP \cdot \sin\alpha} \cdot \frac{EM \cdot MP \cdot \sin\gamma}{FM \cdot MQ \cdot \sin\gamma} \cdot \frac{FM \cdot FQ \cdot \sin\beta}{MD \cdot DP \cdot \sin\beta} \cdot \frac{MD \cdot MP \cdot \sin\delta}{CM \cdot MQ \cdot \sin\delta} = 1.$$

Upon cancellation, multiplication, and rearrangement, it follows that,

$$CQ \cdot FQ \cdot (MP)^2 = EP \cdot DP \cdot (MQ)^2.$$

However, since $CQ \cdot FQ = AQ \cdot QB$ and $EP \cdot DP = BP \cdot AP$, it is true that,

$$AQ \cdot QB \cdot (MP)^2 = BP \cdot AP \cdot (MQ)^2, \text{ or,}$$

$$(a^2 - x^2)y^2 = (a^2 - y^2)x^2.$$

The only positive solution of the above equation is $x = y$, from which it follows that $PM = QM$.

References

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THE BERNSTEIN POLYNOMIALS AND FINITE DIFFERENCES

C. W. GROETSCH and J. T. KING, University of Cincinnati

In a first course in real variables the classical Weierstrass approximation theorem is often proved by considering the sequence of Bernstein polynomials generated by the continuous function f given by

$$B_N(f,\lambda) = \sum_{i=0}^{N} \binom{N}{i} (1-\lambda)^{N-i}\lambda^i f(i/N).$$

Bernstein's proof of the Weierstrass theorem was motivated by the Chebyshev inequality. However, students who are unfamiliar with probability theory usually

Upon cancellation, multiplication, and rearrangement, it follows that,

$$CQ\cdot FQ\cdot (MP)^2 = EP\cdot DP\cdot (MQ)^2.$$

However, since $CQ\cdot FQ = AQ\cdot QB$ and $EP\cdot DP = BP\cdot AP$, it is true that,

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Bernstein's proof of the Weierstrass theorem was motivated by the Chebyshev inequality. However, students who are unfamiliar with probability theory usually fail to appreciate this proof, because there is no apparent reason to suspect that polynomials with this peculiar form will have the desired approximation property.

In this note we hope to motivate the use of the Bernstein polynomials by showing that if $f \in C^2[0,1]$, then the approximating property of these polynomials is a by-product of a standard elementary constructive procedure for approximating the solution of a very simple partial differential equation. Specifically, we define a function U on the triangular region $0 \leqq x + y \leqq 1$ by $U(x,y) = f(x+y)$. It is then clear that U satisfies

$$U_y = U_x; \ U(x,0) = f(x). \tag{1}$$

Since f may also be viewed as the restriction of U to the y-axis we may approximate f by restricting the solution of the forward difference scheme associated with (1) to the y-axis.

Given $\varepsilon > 0$, we choose a positive integer N so that $M/N < \varepsilon$, where

$$M = \sup\{|f''(t)| : t \in [0,1]\}.$$

For any $\lambda \in (0,1]$ we construct an approximate solution u on

$$\{(x,y): 0 \leqq x + y \leqq 1, \ 0 \leqq y \leqq \lambda\}$$

by taking $h = 1/N$, $k = \lambda/N$ and using the naive forward difference scheme:

$$u_{i0} = f(ih), \qquad 0 \leqq i \leqq N \tag{2}$$

$$(u_{i,n+1} - u_{in})/k = (u_{i+1,n} - u_{in})/h \tag{3}$$

for $0 \leqq n \leqq N-1$ and $0 \leqq i \leqq N-n-1$, where $u_{in} \equiv u(ih, nk)$. Since $\lambda = k/h$, equation (3) can be written in the form

$$u_{i,n+1} = ((1-\lambda) + \lambda E)u_{in} \tag{4}$$

where the operator E is defined by $Eu_{in} = u_{i+1,n}$. By use of (4) and (2) we have

$$\begin{aligned} u(0,Nk) &= ((1-\lambda) + \lambda E)^N u(0,0) \\ &= \sum_{i=0}^{N} \binom{N}{i} (1-\lambda)^{N-i}\lambda^i E^i u(0,0) \\ &= B_N(f,\lambda). \end{aligned}$$

It remains now only to estimate $|u(0,Nk) - U(0,Nk)|$. The function U also satisfies (4), save for a truncation error T_{in},

$$U_{i,n+1} = ((1-\lambda) + \lambda E)U_{in} + T_{in}. \tag{5}$$

Solving for T_{in} and applying Taylor's theorem gives

$$\begin{aligned} T_{in} &= kU_y + \frac{k^2}{2}\bar{U}_{yy} - \lambda h U_x - \lambda\frac{h^2}{2}\bar{\bar{U}}_{xx} \\ &= (k^2\bar{U}_{yy} - \lambda h^2 \bar{\bar{U}}_{xx})/2 \end{aligned}$$

where the bars indicate that the derivatives are evaluated at the appropriate intermediate points. Since $k/h = \lambda \leqq 1$ we obtain

$$|T_{in}| \leqq 2h^2M/2 = M/N^2. \tag{6}$$

We denote by e_n the maximum error in the nth row at grid points which contribute to $u(0, Nk)$, i.e.,

$$e_n = \max_{0 \leqq i \leqq N-n} |U_{in} - u_{in}|.$$

Subtracting (5) from (4) we obtain by use of (6)

$$e_{n+1} < e_n + \varepsilon/N.$$

The initial conditions (2) give $e_0 = 0$ and thus

$$e_N = |u(0, Nk) - U(0, Nk)| < N(\varepsilon/N) = \varepsilon.$$

But $u(0, Nk) = B_N(f, \lambda)$ and $U(0, Nk) = U(0, \lambda) = f(\lambda)$, thus $|B_N(f, \lambda) - f(\lambda)| < \varepsilon$ for all $\lambda \in (0, 1]$. Since $B_N(f, 0) = f(0)$, it follows that the Bernstein polynomials approximate f uniformly on $[0, 1]$.

NOTES AND COMMENTS

Eliot William Collins has sent to us three additional solutions to the gunport problem [Vol. 44, No. 4, p. 193; Vol. 45, No 5, p. 280] which he obtained using a short Fortran program. He adds, "In finding these three solutions, only a very small proportion of the possible arrangements of dominoes and blanks on an 8 × 10 board were tested. This leads me to believe that this problem is similar to the knight's tour problem in that solutions, while difficult to find, are very great in number."

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

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Mathematical Thought from Ancient to Modern Times. By Morris Kline. Oxford University, New York, 1972. xvii + 1238 pp. $35.

The author, who has made valuable contributions to applied mathematics, is

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The author, who has made valuable contributions to applied mathematics, is

widely known for his outspoken opposition to the mathematical curricula in the schools. Here he successfully undertakes a formidable project: to survey, at a level comprehensible to a graduate student, the major developments in western mathematics from its beginnings in Mesopotamia until approximately 1930.

At each stage he describes lucidly the motivation for the mathematics, and does not hesitate to explain in adequate detail the mathematics itself. This requires an enviable understanding of mathematics from the most concrete to the most rarified. Contrast this with the many writers who, once they pass the period of the calculus, find themselves in deep water, over their heads, and prefer to talk about mathematics rather than to display some of it; they should put up or shut up.

My objections are few. Let me state them now.

The most regrettable feature of this remarkable book is its price, which puts it out of reach of the students who need it most. Unless a cheaper edition is produced, the book will end up imprisoned by librarians on their reserve shelves. Possibly the writer and publishers would consider an abridged version consisting mainly of mathematics since the Renaissance. After all, histories of mathematics through the Renaissance overflow the library shelves. It is the later period which has lacked adequate treatment. The filling of this gap is the main contribution of Kline's book.

I have one serious quarrel with the writer's choice of what is "major" in pre-1930 mathematics. In spite of his disclaimer in the preface, I believe that *probability* does play an important role in the period in question. Kline gives up the subject after only two paragraphs, ending with Bernoulli's theorem. The vital contributions of Laplace and Gauss to the field are bypassed. The omission is all the more dismaying because of the author's well-known stand on the close relation between mathematics and the study of the physical world. It is familiar to every astronomer that Gauss' use of least squares in orbit determination revolutionized planetary astronomy. And statistical mechanics was important long before Khinchin made it "rigorous" in 1943. But apparently Kline does not consider probability theory important until after 1930. Consequently N. Wiener's name appears only in connection with a forgotten paper on normed linear spaces. The word *statistics* does not appear in the index, nor the name of Bayes.

Another omission. While the work of the Hindus and Arabs receives due attention, that of the Hebrews passes unnoticed. Leonardo of Pisa, the leading Christian mathematician of the Middle Ages, was influenced in his geometric work by that of the Hebrew writer Savasorda, who wrote ca. 1100. For further evidence that the Hebrew contribution is nontrivial see *Studies in Hebrew Astronomy and Mathematics*, by S. Gandz, Ktav Publishing House, New York, 1970.

Another objection. The final paragraph of Chapter 49 (*The Emergence of Abstract Algebra*) begins with the assertion: "However, abstract algebra has subverted its own role in mathematics." and terminates with: "Indeed, most workers in the domain of abstract algebra are no longer aware of the origins of the abstract structures, nor are they concerned with the applications of their results to the concrete fields." While I share the author's private attitude towards mathematics (which is suggested by this choice of words) it is nevertheless unfair to the novice not to point out that there are eloquent defenders of the belief that the study of abstract structures *is* the main object of mathematics.

On p. 615 we find the statement that "the higher-order differentials, which the eighteenth-century men used freely, have not been put on a rigorous basis even today." I believe, on the contrary, that such a theory does exist. It can be found, for example, in Pierpont's classical *Theory of Functions of Real Variables*, I, pp. 276–281. A primitive development, but rigorous, can be found in modern dress in A. Robinson's, *Non-Standard Analysis*, §3.6. If Kline had chosen the word "useful" instead of "rigorous" I should have no quarrel with him. Nor are we told that the collateral concept of the *infinitesimal*, which the 19th century destroyed in its search for rigor, has been restored to a position of both dignity and usefulness; see the book of Robinson just mentioned.

Although the divergence theorem and Stokes' theorem are described, there is not even a passing mention of E. Cartan's exterior differential calculus which makes them both special cases of the same theorem. This calculus is a basic tool of global analysis and differential geometry.

On the other hand, one of the most impressive features of this exposition is the writer's proof that the history of mathematics is part of history itself. For the first half of the book (until ca. 1800; after that time, space prevents an adequate treatment) he describes convincingly the social and religious conditions under which mathematics developed. He does not fall into the trap, common in histories of science, of talking about the relation between science and theology through the Renaissance as "warfare". This doctrine, which we inherit from the period of the Enlightenment, and which my generation was taught as fact, has undergone a thorough reexamination and has been found wanting.

Clearly my objections are small compared to the achievements of the book. I recommend it to all who can afford it, or who can borrow it somewhere.

HARRY POLLARD, Purdue University

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the

there are eloquent defenders of the belief that the study of abstract structures *is* the main object of mathematics.

On p. 615 we find the statement that "the higher-order differentials, which the eighteenth-century men used freely, have not been put on a rigorous basis even today." I believe, on the contrary, that such a theory does exist. It can be found, for example, in Pierpont's classical *Theory of Functions of Real Variables*, I, pp. 276–281. A primitive development, but rigorous, can be found in modern dress in A. Robinson's, *Non-Standard Analysis*, §3.6. If Kline had chosen the word "useful" instead of "rigorous" I should have no quarrel with him. Nor are we told that the collateral concept of the *infinitesimal*, which the 19th century destroyed in its search for rigor, has been restored to a position of both dignity and usefulness; see the book of Robinson just mentioned.

Although the divergence theorem and Stokes' theorem are described, there is not even a passing mention of E. Cartan's exterior differential calculus which makes them both special cases of the same theorem. This calculus is a basic tool of global analysis and differential geometry.

On the other hand, one of the most impressive features of this exposition is the writer's proof that the history of mathematics is part of history itself. For the first half of the book (until ca. 1800; after that time, space prevents an adequate treatment) he describes convincingly the social and religious conditions under which mathematics developed. He does not fall into the trap, common in histories of science, of talking about the relation between science and theology through the Renaissance as "warfare". This doctrine, which we inherit from the period of the Enlightenment, and which my generation was taught as fact, has undergone a thorough reexamination and has been found wanting.

Clearly my objections are small compared to the achievements of the book. I recommend it to all who can afford it, or who can borrow it somewhere.

HARRY POLLARD, Purdue University

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the

talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

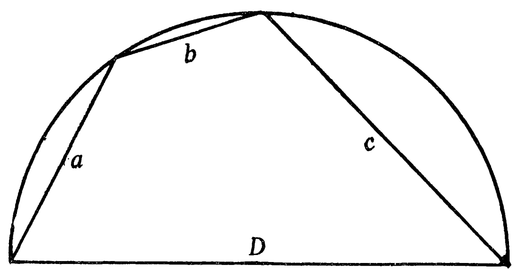
Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before May 1, 1974.

PROPOSALS

880. *Proposed by Richard Corry, Clay Center, Kansas.*

Given chords a, b, c in a semicircle. Determine the diameter (D) of the circle.



881. *Proposed by Raphael T. Coffman, Richland, Washington.*

A root of the equation $x^3 - 2x^2 - 5x - 4 = 0$ correct to two decimal places is 3.66. Show how to find this root using only a series, the first few terms of which are: $1 + 2 + 3 + 20 - 63 + 238 + 871 \cdots$ and show how to generate this series.

882. *Proposed by Charles W. Trigg, San Diego, California.*

One magic square with prime elements is

101	5	71
29	59	89
47	113	17

Form another third order magic square with positive prime elements, six of which are the same as those in the given square.

883.* *Proposed by Harry Pollard, Purdue University.*

Prove that if the sequence $\{N_n\}$ satisfies the recursion

$$1 + N_{n+1}^{-3} = \frac{9(N_n^2 - N_n + 1)}{(N_n - 2)^3}$$

and $N_n > 2$ then the sequence is constant.

884. *Proposed by K. W. Schmidt, University of Manitoba.*

A sine curve, parallel to the x-axis, has between a peak and an adjacent trough 4 points with equidistant x-components. Given the space between the x-components, and any 2 of the differences between the y-components, find the shortest possible expression for the frequency.

885. *Proposed by Stephen B. Maurer, Phillips Exeter Academy.*

Let Z_n be an additive group of integers modulo n. For what values of n do there exist permutations $f, g\colon Z_n \to Z_n$ such that $f + g$ is a permutation also?

886.* *Proposed by Doug Engel, Denver, Colorado.*

In a sequence of positive integers, $N_{k+1} = N_k +$ the sum of all the distinct prime factors of N_k including 1 and N_k, $k = 0, 1, 2, \cdots$. Such a sequence is 1, 2, 3, 4, 7, 8, 11, 12, 18, 24 $\cdots$. The sequence 5, 6, 12, 18 $\cdots$ merges with the first sequence as do all sequences with $N_0 < 91$.

It is conjectured that such sequences with N_0 a positive integer merge with the basic sequence that begins with 1. Prove or disprove this conjecture. If disproven, how many independent sequences exist?

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q578. Prove that the area of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose coordinates are integers has an area N or $N/2$ for some positive integer N.

[*Submitted by Marvin L. Bittinger*]

Q579. For each real number x prove that

$$-x^n + \sum_{k=0}^{2n} (-1)^k x^k \geqq 0.$$

[*Submitted by Erwin Just*]

Q580. Determine the extreme values of

$$\frac{2\sin A\cos B\cos C + 2\sin B\cos C\cos A + 2\sin C\cos A\cos B}{\sin 2A + \sin 2B + \sin 2C}$$

where A, B, C denote the angles of a triangle.

[*Submitted by Murray S. Klamkin*]

Q581. The inequality $|E(xy)| \leqq |E(|x|y)|$ certainly "looks true". Is it true for any pair of random variables?

[*Submitted by B. B. Winter*]

Q582. Prove that if $a > 0$, $b > 0$ and $c > 0$, then $a^3 + b^3 + c^3 \geqq 3abc$.

[*Submitted by Steven R. Conrad*]

(Answers on pages 296–297)

SOLUTIONS

Late Solutions

Ranee Gupta, University-Leggett School, Michigan, **851**; *Andrzej Makowski, Institute of Mathematics, Warsaw, Poland,* **845**; *Robert L. Stacy, Manzano High School, Albuquerque, New Mexico,* **838, 839.**

Checkerboard Covering

852. [January, 1973] *Proposed by Richard A. Gibbs, Fort Lewis College, Colorado.*

If a $2k \times 2k$ checkerboard is given with two diagonal corners removed, it is well known that it cannot be covered by 2×1 dominoes because the removed corners are of the same color and a domino must cover one square of each color. (a) Is it possible to cover a $2k \times 2k$ checkerboard which has had one square of each color removed? (b) What about a $(2k + 1) \times (2k + 1)$ checkerboard which has had one square of the corner color removed?

I. *Solution by Michael Gilpin, Michigan Technological University.*

Let P_k be the statement that a $2k \times 2k$ checkerboard having no squares removed or having one square of each color removed can be covered by 2×1 dominoes. Clearly P_1 holds. Assume then that P_n holds and let Δ be a suitable $(2n + 2) \times (2n+2)$ checkerboard. If B denotes the squares on the perimeter of Δ, then Δ is the union of B and a $2n \times 2n$ checkerboard Δ'. If 0 or 2 squares have been removed from B, then one can tile B completely without covering squares of Δ'. If (say) one white square of B is missing, then one can tile B with $4k + 2$ dominoes with one domino also covering a white square of Δ'. In either case the induction hypothesis P_n applies to the remaining squares of Δ'. Thus the answer to (a) is yes.

The answer to (b) is also yes. A $(2k + 3) \times (2k + 3)$ checkerboard having one corner square removed is the disjoint union of a $2 \times (2k + 3)$ rectangle, a $2 \times (2k + 1)$ rectangle and a $(2k + 1) \times (2k + 1)$ checkerboard having one corner square removed. The case $k = 1$ is clear and the above observation gives the inductive step.

II. *Solution by Charles W. Trigg, San Diego, California.*

(a) It is only necessary to partition the checkerboard into a closed path one square wide. This can be done in a great many ways. The path established by the heavy lines in Figure 1, due to Ralph E. Gomory, was first given in Martin Gardner's *Mathematical Games* column on Page 152, of the November, 1962, Scientific American. Three other designs of this type appear on Page 96 of my book, *Mathematical Quickies* (1967).

The squares lie with alternating colors along the closed path. The removal of two squares of opposite colors from any two positions along the path will cut the path into two open-ended segments (or one segment if the removed squares are adjacent on the path). Each segment must consist of an even number of squares, so each segment must be completely covered by dominoes. Clearly, the design and the argument can be extended to any $2k \times 2k$ checkerboard.

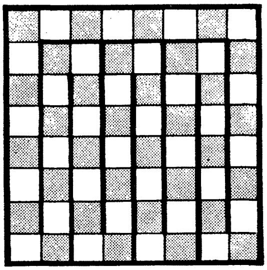


FIG. 1

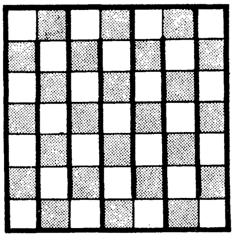


FIG. 2

(b) On a $(2k+1) \times (2k+1)$ checkerboard, the covering path may be open-ended as in Figure 2. Removal of a square of the corner color will divide the path into two segments (or one, if the path-end square is removed) each with an even number of squares of alternating colors. Consequently, they can be covered with dominoes.

Also solved by Joel Brazovsky, Whitman College, Washington; Frank M. Eccles, Phillips Academy, Andover, Massachusetts; Ralph E. Edwards, Baltimore Life Insurance Company, Maryland; Michael Goldberg, Washington, D.C.; Jacob F. Golightly, Jacksonville University, Florida; John M. Howell, Littlerock, California; Ralph Jones, University of Massachusetts; Murray S. Klamkin, Ford Motor Company, Michigan; Vaclav Konecny, Hawkins, Texas; John Oman and Andrew L. Perrie (jointly), University of Wisconsin at Oshkosh; Phil Tracy, Liverpool, New York; and the proposer.

An Eight Point Problem

853. [January, 1973] *Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.*

It is a well-known theorem that all quadric surfaces which pass through seven given points will also pass through an eighth fixed point. (a) If the seven given points are (0,0,0), (0,0,1), (0,1,0), (2,0,0), (1,1,0), (1,0, 1) and (1,1,1), determine the eighth fixed point. (b) Determine the eighth fixed point explicitly as a function of the seven general given points (x_i, y_i, z_i), $i = 1,2,3,\cdots 7$.

Solution by the proposer.

If the equation of a quadric surface be

$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gxhy + iz + j = 0,$$

then the coefficients must satisfy the 7 equations

$$j = 0, \qquad a + b + d + g + h = 0,$$

$$c + i = 0, \qquad a + c + f + g + i = 0,$$

$$b + h = 0, \qquad a + b + c + d + e + f + g + h + i = 0$$

$$4a + 2g = 0.$$

Thus, the equation reduces to the form

$$a(x^2 + xy - zy + zx - 2x) + by(y - 1) + cz(z - 1) = 0$$

and the eighth fixed point is $(-1,1,1)$.

A Partially Composite Function

854. [January, 1973] *Proposed by John D. Baum, Oberlin College, Ohio.*

Let $N = x^4 + 4a^4$ where x and a are integers, then N is composite unless $x = a = \pm 1$.

Solution by Robert S. Stacy, Manzano High School, Albuquerque, New Mexico.

$x^4 + 4a^4 = x^4 + 4a^2x^2 + 4a^4 - 4a^2x^2 = (x^2 + 2a^2 - 2ax)(x^2 + 2a^2 + 2ax) = [(x - a)^2 + a^2][(x + a)^2 + a^2]$. If x and a are positive integers and if N is a prime, $x - a < x + a$ so that $(x - a)^2 + a^2 = 1$. This is possible only if $(x - a)^2 = 0$ and $a^2 = 1$. That is, $x = a = \pm 1$.

(The theorem does not hold for nonpositive a and x. For $x = 0$ and $a = 1, N = 1$ which is not composite. If (k, k) is a solution, $(k, -k)$ is also in which case $x \neq a$.)

Also solved by Joe Albree, Louisiana State University, New Orleans; Jose C. Antunes, Fort Lee, Virginia; Leon Bankoff, Los Angeles, California; Marion T. Bird, California State University, San Jose; Charles K. Brown, III, Westtown, Pennsylvania; Steven R. Conrad. B. M. Cardozo High School, New York; Romae J. Cormier, Northern Illinois University; Miltiades S. Demos, Villanova University; Santo M. Diano, Havertown, Pennsylvania; Ralph E. Edwards, Baltimore Life Insurance Company, Maryland; Abraham L. Epstein, Hanscom Field, Massachusetts; Marjorie Fitting, California State University, San Jose; William F. Fox, Moberly Junior College, Missouri; Leon Gerber, St. Johns University; Jacob F. Golightly, Jacksonville University, Florida; Ranee Gupta, University — Leggett

School, Michigan; Ann Goodsell, University of Wisconsin-Oshkosh; M. G. Greening, University of New South Wales, Australia; Steve Hartman, The New Lincoln School, New York, Karl Heuer and G. A. Heuer (jointly), Moorhead Minnesota; Howard Lyn Hiller, Cornell University; Larry Hoehn, Knovxille, Tennessee: B. G. Hoerbelt, Genesee Community College, New York; J. A. H. Hunter, Toronto, Canada; Eric Jorgensen and P. D. Vestergaard (jointly), Danmarks Ingeniorakademi, Denmark; Irwin Just, Bronx Community College; Richard Kerns, Hamburg, Germany; Murray S. Klamkin, Ford Motor Company, Michigan; Vaclav Konecny, Hawkins, Texas; Lew Kowarski, Morgan State College, Maryland; Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston, Massachusetts; M. Renn Murty and V. Kumar Murty (jointly), Ottawa, Canada; Number Theory Class, University of Georgia; F. D. Parker, St. Lawrence University, New York; Cecil G. Phipps, Cookeville, Tennessee; Gordon D. Prichett, Hamilton College, New York; Bob Prielipp, University of Wisconsin — Oshkosh; Kenneth Rosen, Massachusetts Institute of Technology; Gabriel Rosenberg, Pace College, New York; Harry D. Ruderman, Hunter College, New York; John Rueter, Augustana College, South Dakota; Ben Sapolsky, Bok Vocational Technical School, Philadelphia, Pennsylvania; Joseph Silverman, White Plains, New York; Phil Tracy, Liverpool, New York; Charles W. Trigg, San Diego, California; C. S. Venkataraman, Trichur, India; John C. Wenger, The Loop College, Chicago, Illinois; William Wernick, City College, New York; Kenneth M. Wilke, Topeka Kansas; K. L. Yocom, South Dakota State University; Gene Zirkel, Nassau Community College, New York; and the proposer. One unsigned solution was received.

A Particular Pythagorean Triangle

855. [January, 1973] *Proposed by Romae J. Cormier, Northern Illinois University, De Kalb, Illinois.*

Let T and T' be two Pythagorean triangles. If θ and θ' are any acute angles of these triangles respectively such that $\theta \neq \theta'$, then show that the right triangle T'' which has an acute angle $\theta + \theta'$ or $\pi - \theta - \theta'$ is Pythagorean.

I. *Solution by Hubert J. Ludwig, Ball State University, Indiana.*

Let a, b, c, x, y, z be positive integers. Let right triangle T have legs a and b, hypotenuse c, and let the acute angle opposite the leg of length a be θ. Let right triangle T' have legs x and y, hypotenuse z, and let the acute angle opposite the leg of length x be θ'.

Then $\sin(\theta + \theta') = (ay + bx)/cz$, $\cos(\theta + \theta') = (by - ax)/cz$, and if $by - ax > 0$, then $(ay + bx, by - ax, cz)$ is a Pythagorean triple.

If, however, $by - ax < 0$, we then have $\sin\{\pi - (\theta + \theta')\} = (ay + bx)cz$, $\cos\{\pi - (\theta + \theta') = (ax - by)/cz$, and then $(ay + bx, ax - by, cz)$ is a Pythagorean triple.

The restriction that $\theta \neq \theta'$ is not necessary; we merely need the requirement that $\theta + \theta' \neq \pi/2$. For example, if we consider triangle T with $b \neq a$, we then have $\sin 2\theta = (2ab)/c^2$ and $\cos 2\theta = (b^2 - a^2)/c^2$. If $b^2 - a^2 < 0$ then $(2ab, b^2 - a^2, c^2)$ is the required Pythagorean triple, when we have $b^2 - a^2 < 0$ we have $(2ab, a^2 - b^2, c^2)$ as the desired Pythagorean triple.

Additionally, it is easy to show that given the triangles T and T' there is a triangle T'' which has an acute angle $\theta - \theta'$ or $(\pi/2) + (\theta - \theta')$ which is Pythagorean. In this case we have $\sin(\theta - \theta') = (ay - bx)/cz$ and $\cos(\theta - \theta') = (by + ax)/cz$ and if

$ay - bx > 0$ then $(ay - bx, by + ax, cz)$ is a Pythagorean triple. If $ay - bx < 0$ then $\sin\{(\pi/2) + (\theta - \theta')\} = (by + ax)/cz$, $\cos\{(\pi/2) + (\theta - \theta') = (bx - ay)/cz$ and $(by - ax, bx - ay, cz)$ is a Pythagorean triple. In this case we do need the restriction $\theta \neq \theta'$.

II. *Solution by Harry D. Ruderman, Hunter College High School, New York.*

Let the sides of triangle T be a, b, c so that $a^2 + b^2 = c^2$.

Let the sides of triangle T' be r, s, t so that $r^2 + s^2 = t^2$.

Let θ be opposite a so that $\tan \theta = (a/b)$ and let θ' be opposite r so that $\tan \theta' = r/s$.

Then $\tan(\theta + \theta') = (as + br)/(bs - ar)$. If $\theta + \theta' < 90°$ the sides of a Pythagorean triangle are $as + br$, $bs - ar$, ct because $(as + br)^2 + (bs - ar)^2 = a^2s^2 + b^2r^2 + b^2s^2 + a^2r^2 = (a^2 + b^2)(r^2 + s^2) = c^2t^2$.

If $\theta + \theta' > 90°$ then the sides are $as + br$, $ar - bs$, ct and an angle of the Pythagorean triangle could be $180° - \theta - \theta'$.

Note: If $\theta > \theta'$ then another set of sides for a Pythagorean triangle could be $as - br$, $bs + ar, ct$. To summarize:

$$|as \pm br|, |ar \mp bs|, ct$$

will be sides for a Pythagorean triangle, $\theta - \theta'$ being an angle of a Pythagorean triangle.

Also solved by M. Ahuja, Southeast Missouri State College; Leon Bankoff, Los Angeles, California; Merrill Barnebey, University of Wisconsin, LaCrosse; Gladwin Bartel, LaJunta, Colorado; Gerald E. Bergum, South Dakota State University; Charles K. Brown, III, Westtown, Pennsylvania; Santo M. Diano, Havertown, Pennsylvania; Ralph Garfield, College of Insurance, New York; Michael Goldberg, Washington, D.C.; Jacob F. Golightly, Northern Illinois University; M. G. Greening, University of New South Wales, Australia; Ranee Gupta, University-Leggett School, Michigan; Karl Heuer, Moorhead High School, Moorhead, Minnesota; Ralph Jones, Northern Illinois University; Vaclav Konecny, Hawkins, Texas; Lew Kowarski, Morgan State College, Maryland; Robert Larson; Henry S. Lieberman, John Hancock Mutual Life Insurance Company, Boston, Massachusetts; F. D. Parker, St. Lawrence University, New York; Gordon D. Prichett, Hamilton College; Ben Sapolsky, Bok Vocational Technical School, Philadelphia, Pennsylvania; Robert S. Stacy, Albuquerque, New Mexico; Phil Tracy, Liverpool, New York; Charles W. Trigg, San Diego, California; C. S. Venkataraman, Trichur, India; Lance Wheeler; Kenneth M. Wilke, Topeka, Kansas; K. L. Yocom, South Dakota State University; Gene Zirkel, Nassau Community College, New York; and the proposer. One incorrect solution was received.

A Logarithmic Inequality

856.* [January, 1973] *Proposed by Leonard Gallagher, University of Colorado.*

For what values of x and y is the following logarithmic inequality valid:

$$\frac{\log^2 x \log^2 y}{\log^2 x + \log^2 y} < \log^2 \left(\frac{xy}{x+y}\right).$$

Solution by M. T. Bird, California State University, San Jose.

We assume $x > 0$ and $y > 0$. We must exclude the point $x = y = 1$ and all points of the hyperbolic arc $C_1 : xy = x + y,\ x > 1$. The points $x = 1$ and $y \neq 1$ and the points $x \neq 1$ and $y = 1$ satisfy the given identity. If $x < 1$ we may introduce $u = 1/x$ and $v = 1/y$ so that the given inequality has the form

$$\frac{\log^2 u \log^2 v}{\log^2 u + \log^2 v} < \log^2(u + v),\ u > 1,\ v > 0.$$

Under these conditions $\log^2 u$ dominates the left member and is dominated by the right member. We see that the given inequality is satisfied if $x < 1$. Similarly we find the inequality is satisfied if $y < 1$.

We consider $x > 1$ and $y > 1$. We designate the portions of this region below and above the hyperbolic arc C_1 as R_2 and R_3, respectively. If we introduce $t = \log y/\log x$ the given inequality has the form

$$t^2 \log^2 x/(1 + t^2) < \log^2[x^t/(1 + x^{t-1})].$$

For all points in R_2 the given inequality may be replaced by

$$t \log x/\sqrt{1 + t^2} < \log(1 + x^{t-1}) - t \log x$$

or

$$t(1 + \sqrt{1 + t^2}) \log x < \sqrt{1 + t^2} \log(1 + x^{t-1}).$$

For $t > 0$ there is a unique point $P_2 = (x_2, y_2)$ whose coordinates satisfy the equations

$$t(1 + \sqrt{1 + t^2}) \log x = \sqrt{1 + t^2} \log(1 + x^{t-1}),\ y = x^t.$$

All points in R_2 below P_2 satisfy the given inequality. We designate as C_2 the set of points P_2 for which $t > 0$.

Similarly, for all points in R_3 the given inequality may be replaced by

$$t \log x/\sqrt{1 + t^2} < t \log x - \log(1 + x^{t-1})$$

or

$$\sqrt{1 + t^2} \log(1 + x^{t-1}) < t(\sqrt{1 + t^2} - 1) \log x.$$

For $t > 0$ there is a unique point $P_3 = (x_3, y_3)$ whose coordinates satisfy the equations

$$\sqrt{1 + t^2} \log(1 + x^{t-1}) = t(\sqrt{1 + t^2} - 1) \log x,\ y = x^t.$$

All points in R_3 above P_3 satisfy the given inequality. We designate as C_3 the set of points P_3 for which $t > 0$.

We conclude that the given inequality is satisfied by all points for which $x > 0$

and $y > 0$ except $x = y = 1$ and the points which are both on or above C_2 and on or below C_3.

A Boundary Value Problem

857. [January, 1973] *Proposed by Marlow Sholander, Case Western Reserve University.*

Find a function $y = f(x)$ such that:
(i) $f(0) = 2, f(2) = 1, f(4) = 2$.
(ii) $f(x)$ is differentiable on $0 \leqq x \leqq 4$.
(iii) There is an area-length equality

$$\int_0^x y\,dx = \int_0^x \sqrt{1+(y')^2}\,dx.$$

I. *Solution by Harley Flanders, Tel Aviv University, Israel.*

The solutions of

$y = \sqrt{1+(y')^2}$, or $y^2 = 1 + (y')^2$ are $y = 1$ and $y = \frac{1}{2}(Ae^x + A^{-1}e^{-x})$ if $y > 1$. An interpolation is necessary:

$$y = \begin{cases} \frac{1}{2}(Ae^x + A^{-1}e^{-x}) & 0 \leqq x \leqq -\ln A \\ 1 & -\ln A \leqq x \leqq -\ln B \\ \frac{1}{2}(Be^x + B^{-1}e^{-x}) & -\ln B \leqq x \leqq 4 \end{cases}$$

where $A = 2 - \sqrt{3}$, $B = (2+\sqrt{3})e^{-4}$.

This is an instructive problem in a less than obvious boundary value problem.

II. *Solution by Benjamin L. Schwartz, McLean, Virginia.*

Solving the differential equation $y = \sqrt{1+(y')^2}$ $(y > 0)$ we get $y = \cosh(x + C)$ as the general solution; but there is a singular solution, $y = 1$. The boundary conditions of the problem can be met only by combining segments of two particular solutions and the singular solution as follows:

$$y = \cosh(x-k) \text{ for } 0 \leqq x \leqq k,$$

$$y = 1 \text{ for } k < x \leqq 4 - k$$

$$y = \cosh(x - 4 + k) \text{ for } 4 - c < x,$$

where $k = \cosh^{-1} 2 \doteq 1.32$.

Also solved by V. Chinnaswamy, Williamsport Area Community College, Pennsylvania; Miltiades S. Demos, Villanova University; Walter O. Egerland, Aberdeen Proving Ground, Maryland; The Eisenhower College Problem Solving Group, Seneca, New York; U. Hartman, E. Jorgensen and P. D. Vestergaard (jointly), Denmarks Ingeniorakedmi, Denmark; Mel Harvey, Fort Lewis College, Colorado; G. A. Heuer and Karl Heuer (jointly), Moorhead, Minnesota; Ralph Jones, University of Massachusetts, *Amherst; Emil F. Knapp, Augustana College, South Dakota; Vaclav Konecny, Hawkins, Texas; John Oman and Andrew L. Perrie (jointly,) University of Wisconsin-Oshkosh; Cecil G. Phipps, Tennessee Technological University; Joseph Silverman, White Plains, New York; Wolfe Snow, Brooklyn College, New York; Robert S. Stacy, Albuquerque, New Mexico; Phil Tracy, Liverpool, New York; Paul J. Zwier, Calvin College; and the proposer. One unsigned solution was received.*

Probability of an Obtuse Triangle

858. [January, 1973] *Proposed by David Singmaster, Polytechnic of the South Bank, London, England.*

In the November 1970 issue, Eric Langford has obtained the probability that three points chosen at random in a $1 \times L$ rectangle form an obtuse triangle. Consider two points chosen at random in the unit interval. It is not difficult to see that the probability of the three segments forming a triangle is $\frac{1}{4}$. What is the probabiilty that the resulting triangle is obtuse?

Solution by Major G. C. Holterman, USA, Petersberg, Virginia.

Let X denote a random variable giving the fractional length of the segment to the left of one of the points, and let Y denote a random variable giving the fractional length to the left of the other point selected. The statement of the problem justifies the assumption that both X and Y are uniformly distributed on the unit interval. It is not too difficult to see that the probability that the three resulting parts of the segment can be arranged to form a triangle is 0.25. Those points $(X = x, Y = y)$ for which a triangle can be formed are shown by the unshaded areas in Figure 1.

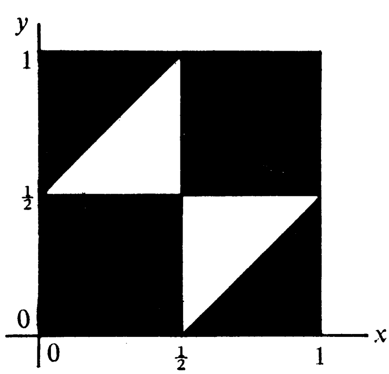


FIG. 1

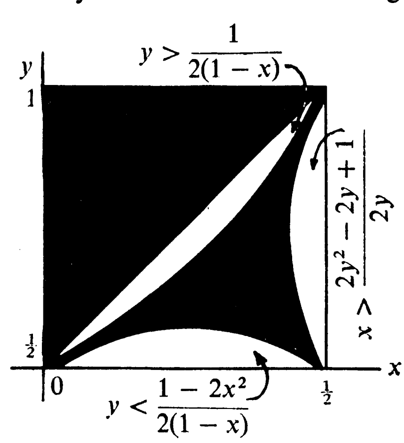


FIG. 2

Consider the points $(x < \frac{1}{2},\ y < x + \frac{1}{2})$. Let r denote the length of the shortest side of the triangle determined by some (x, y) in this area. Let s and t denote the lengths of the other two sides. The triangle is obtuse if and only if either of the following conditions is satisfied:

$$\text{A.}\quad S^2 > r^2 + t^2$$

$$\text{B.}\quad t^2 > r^2 + s^2.$$

Case I. For $y \leqq 1 - x$, we have: $r = x$; $s = 1 - y$; $s = 1 - y$; and $t = y - x$. Applying A and B, the triangle is obtuse if and only if either $y < (1 - 2x^2)/2(1-x)$ or $y > 1/2(1-x)$.

Case II. For $y \geqq 1 - x$, we have: $r = 1 - y$; $s = x$; and $t = y - x$. Applying A and B, the triangle is obtuse if and only if either $x > (2y^2 - 2y + 1)/2y$ or $y > 1/2(1 - x)$.

The shaded area of Figure 2 shows those points under consideration for which an obtuse triangle is not possible. The cases for $x > \frac{1}{2}$ are analogous. Hence, we have

$$P_1 = 2\left[\left(3/8 - \int_0^{\frac{1}{2}} \frac{dx}{2(1-x)}\right) + \left(\int_0^{\frac{1}{2}} \frac{1-2x^2}{2(1-x)} dx - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \int_{\frac{1}{2}}^{1} \frac{2y^2 - 2y + 1}{2y} dy\right)\right] = 3 \ln \tfrac{1}{2} + 9/4 \approx 0.17.$$

If we are given that a triangle can be formed, the probability that the triangle is obtuse is then given by:

$$P_2 = P_1/0.25 \approx 0.68.$$

Also solved by Michael Goldberg, Washington, D. C.; Jacob F. Golightly, Jacksonville University, Florida; H. Martin Hemit, Reedly High School, California; Karl Heuer and G. A. Heuer (jointly), Moorhead, Minnesota; Vaclav Konecny, Hawkins, Texas; and the proposer. Murray S. Klamkin, Ford Motor Company, found the problem in J. Edwards, *A Treatise On Integral Calculus*, 1954, pp. 810–811. Steven R. Conrad found in Ross Honsberger, *Ingenuity In Mathematics*, a problem "find the probability that $(x,y,1)$ with x and y in $(0,1)$ are the sides of an obtuse triangle." The solution given is $(\pi - 2)/4$. Conrad found a second reference in Yaglom and Yaglom, *Challenging Mathematical Problems With Elementary Solutions* (Vol. 2), page 32, Problem 155.

Comment on Problem 837

837. [May, 1972, and March, 1973] *Proposed by Vladmir F. Ivanoff, San Carlos, California.*

Prove that the altitudes of any triangle bisect the angles of another triangle whose vertices are the feet of the altitudes of the first triangle.

Comment by Julius G. Baron, Rye, New York.

There is a simple planimetric solution of the problem.

Let A', B' and C' be the feet of the altitudes of the triangle ABC. Draw half-circles on points $AC'A'C$ and $BC'B'C$.

$$\measuredangle CBB' = 90° - \measuredangle ACB = \measuredangle CAA'.$$

Also,

$$\measuredangle CC'A' = \measuredangle CAA', \text{ and } \measuredangle CC'B' = \measuredangle CBB'.$$

Therefore,

$$\measuredangle CC'B' = \measuredangle CC'A'.$$

Comment on Q562

Q562. [March, 1973] Let $f(z)$ be a bounded entire function. Then $f(z)$ is a constant.

[*Submitted by Warren Page*]

Comment by C. F. Pinzka, University of Cincinnati.

It is not true that if $M_n = \max_{|z|=n}|f(z)|$ is strictly increasing then $M \to \infty$ as $n \to \infty$. $M_n = 1 - 1/n$ is a simple example, while $f(z) = 1 - 1/z$ is an example of a function bounded away from the origin whose maximum modulus on the circle $|z| = n$ increases strictly with n. The trouble is that $f(z)$ is not analytic at the origin and thus not entire.

A proof of Liouville's theorem can be constructed by using Cauchy's inequality in the form $|f'(z)| \leqq M/n$, where M is an upper bound of $|f(z)|$, from which it follows that $f'(z) = 0$ and thus $f(z)$ is constant.

ANSWERS

A578. Let

$$D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

then the area of the triangle is $\frac{1}{2}|D|$. Now if the coordinates are all integers, then D is an integer, so $N = |D|$.

A579. The arithmetic-geometric mean inequality guarantees that for each $x, x^{2k} + x^{2k+2} \geqq 2x^{2k+1}$. Therefore, $\Sigma_{k=0}^{n-1}(x^{2k} + x^{2k+2}) \geqq 2\Sigma_{k=0}^{n-1}x^{2k+1}$ and this implies $1 + x^{2n} + 2\Sigma_{k=1}^{n-1}x^{2k} \geqq 2\Sigma_{k=0}^{n-1}x^{2k+1}$. Adding $1 + x^{2n} \geqq 2x^n$ to the latter inequality yields

$$2\Sigma_{k=0}^{n}x^{2k} \geqq 2[x^n + \Sigma_{k=0}^{n-1}x^{2k+1}] \text{ or } \Sigma_{k=0}^{n}x^{2k} \geqq x^n + \Sigma_{k=0}^{n-1}x^{2k+1}$$

or $-x^n + \Sigma_{k=0}^{2n}(-1)^k x^k \geqq 0$.

A580. The expression is equivalent to

$$\frac{\Sigma\ (a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}{\Sigma\ a^2(b^2 + c^2 - a^2)} = 1.$$

A581. Suppose that you win or lose \$1.00, depending on whether a tossed fair coin lands Heads or Tails. If the r.v. X represents your net gain, and $Y = -X$, then $[E(XY)] > [E(|X|Y)]$.

A582. Let $x = a^3$, $y = b^3$ and $z = c^3$. The original inequality then becomes

$$\frac{x + y + z}{3} \geqq \sqrt[3]{xyz}$$

which is the familiar arithmetic geometric mean problem. This derived inequality is certainly true, hence the original inequality is clearly true.

(Quickies on pages 286–287)

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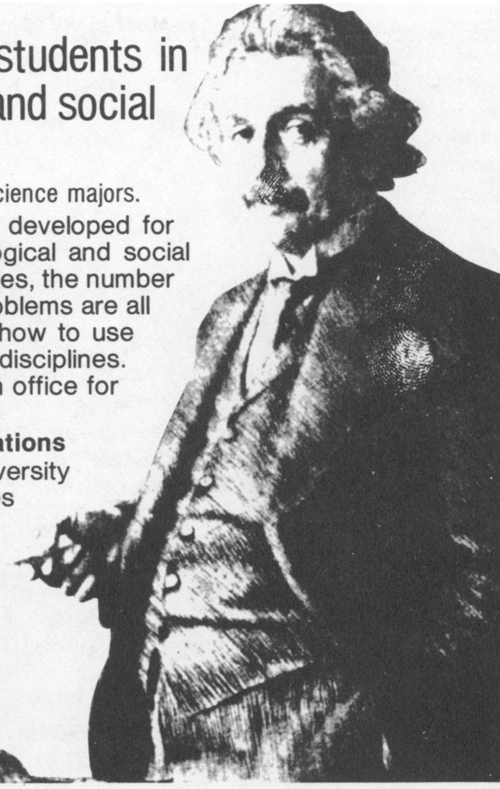

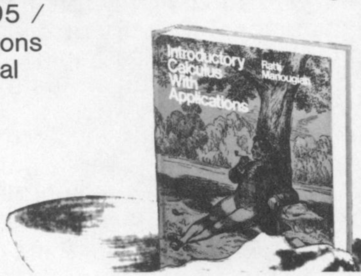

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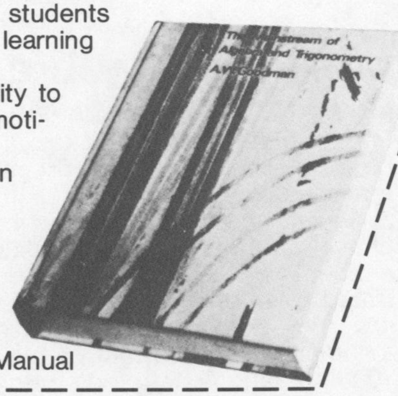

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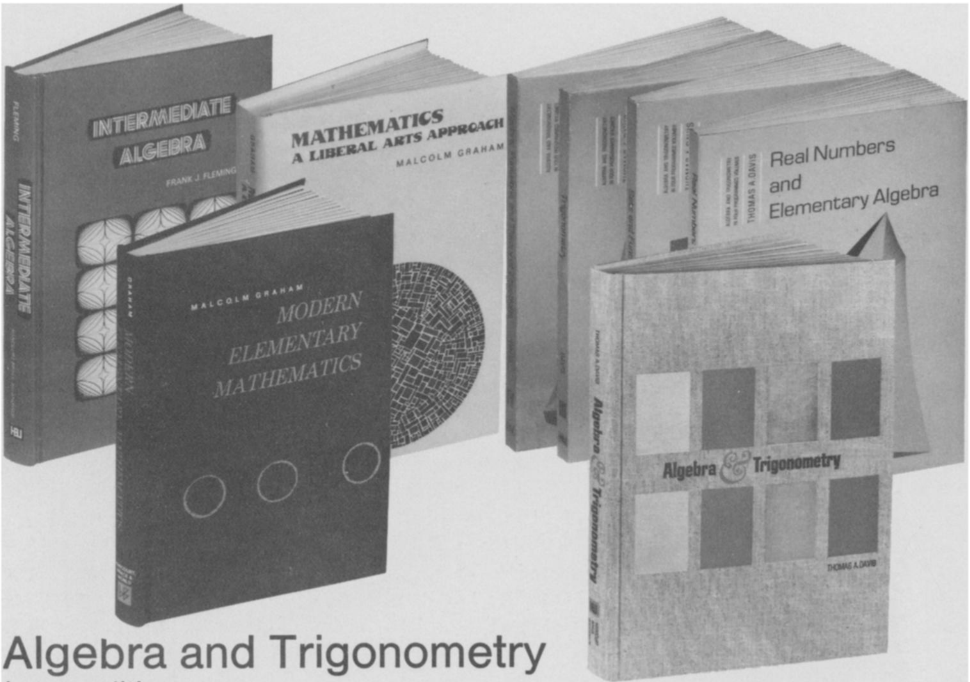

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